Optimal Control of a Medical Waiting System with Autonomous Patients

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Abstract

Waiting systems for medical services have two distinctive features that differentiate them from queueing systems commonly studied in Operations Management: variability in the quality of service, and patient autonomy. That is, patients in these systems recognize that waiting has an option value and, hence, are willing to decline a service offer if they expect future offers to be more attractive. In order to capture this behavior and identify mechanisms that alleviate the resulting externalities, we develop and analyze a stylized competitive queueing model. The model is an M/M/1 queue with reneging with the additional assumptions that patients may decline a service offer in order to maximize their own welfare, and that the quality of service is random and highly variable. Under an assumption of perfect and complete information we show that the queueing discipline is a potent instrument that can be used by the medical planner overseeing the waiting system in order to maximize social welfare. In particular, while the commonly used First-Come First-Serve (FCFS) discipline is found to be inefficient, Last-Come First-Serve (LCFS) achieves the socially optimal outcome. A numerical example demonstrates that the welfare losses from FCFS can be substantial.

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1 Introduction

Wait lists are common both in publicly-funded and in privately-funded health care systems. In United Kingdom’s National Health Services (NHS), the wait for some elective procedures is now more than a year long (Economist, 2001). Even in the United States where market forces are prevalent, long waits are not uncommon. A recent Wall Street Journal article states that women in some states may have to wait six months or more for a mammogram (WSJ, 2000). The most dramatic example of a medical wait list is provided by organ transplantation where the continued shortage of organs generates waits in excess of three years (UNOS, 2002).

These medical waiting systems share many features in common with queuing systems commonly studied in Operations Management (OM). However, unlike the standard OM models that are primarily concerned with variability in the duration of service, medical systems are also plagued by variability in the quality of “service” (i.e., treatment) offered. Further, patients in these systems are autonomous and, hence, they may decline a “service” offer in anticipation of a potentially superior future offer. For example, in organ allocation there is substantial variability in the type and quality of organs that become available for transplantation (the “service”) and patients may decline an offer for an organ of poor quality in anticipation of a future offer of better quality. It is our premise that these two features of medical waiting systems, that is quality variability and patient autonomy, generate important challenges, and their study is the objective of this manuscript.

In order to set the stage for our analysis, it is instructive to first outline the main effects that these two features have on the performance of a medical waiting list. Consider the organ transplantation system. When a patient declines an organ offer, then an effort must be made to identify another candidate willing to accept that organ. However, the organ accumulates “cold-ischemia time” (i.e. time that the donor organ spends outside a host) during this time-consuming search, and this degrades the quality of the outcomes from the transplant operation. Further, it is possible that an organ may eventually be discarded thus, exacerbating the organ shortage. Another interesting backdrop for our study is provided by nursing home waiting lists. Results presented in Shapiro et al. (1992) demonstrate that patient choice contributes to longer waiting times. The same authors suggest that a policy
in which patients are placed in an interim facility pending transfer to a facility of their choice can lead to a considerable reduction in waiting times.

Medical authorities managing these waiting systems have two important levers they can use in order to reduce the detrimental effects of patient autonomy: *rationing* and *prioritization*. Rationing restricts patient access to certain treatment options while prioritization ranks patients on the waiting list to determine who will be offered each available treatment option. The question is which of the two mechanisms is more effective in preventing socially inefficient behavior by the patients on the waiting list. An exploration of each option in isolation can easily provide a rationale for its utility in preventing such behavior. For example, a dynamic rationing mechanism may reduce a patient’s incentive to decline an offer if such a decline will be followed by a reduction in the patient’s future range of treatment options (i.e. a form of punitive rationing). Similarly, a priority system in which a patient’s waiting-list ranking drops after a treatment is declined, is likely to achieve the desired outcome. However, is it necessary for the medical authorities to utilize both levers, or is one of them sufficient to induce the desired behavior?

In this paper, we develop and analyze a novel queuing model that provides an answer to this question. Our model is simple and stylized but captures the two important issues that are the essence of the problem: treatment variability and patient autonomy. In this model, patients arrive according to a time-homogeneous Poisson process and join a single server queue. Treatment opportunities arrive with i.i.d. exponential interarrival intervals, and each treatment option is associated with a quality reward that reflects a patient’s quality adjusted life expectancy after treatment. The quality reward is revealed upon arrival of the treatment opportunity and is observed by all parties. Patients on the waiting list also receive a reward per unit time that reflects their quality of life prior to treatment. Patients may renege from this system with a constant renegoting rate that reflects patient death or migration. Each patient aspires to maximize her own total expected discounted reward, and the medical authority managing the system (referred to as the social or medical planner) wishes to maximize the total infinite horizon expected discounted reward for all patients participating in the waiting system. To achieve their respective goals, patients may decline treatment offers while the medical authorities may ration the treatment options they offer.
to the patients in a dynamic fashion, or may select appropriate dynamic priority rules.

The first step in our analysis identifies the socially efficient (or pareto optimal) outcome for this medical waiting list. This involves an analysis of a frictionless ideal where patients are not-autonomous; that is, they always accept the treatment offered to them. In this case, patient prioritization is not meaningful since all patients are identical, but treatment rationing is effective because there is a trade-off between waiting and accepting a “not-so-good” treatment. In fact, it is shown that as the queue length increases, the range of treatment options made available to the patients should also increase.

The next step is to consider the competitive equilibria that will emerge when patients are autonomous. We start with the assumption that patients are ranked according to the first-come first-served (FCFS) discipline, and we investigate whether, given this priority discipline, treatment rationing increases the planner’s objective. To our surprise we discover that rationing has no effect since it is confounded by patient autonomy. In fact, the rationing rule that maximizes the planner’s objective coincides with a rationing rule that emerges as the competitive equilibrium when the central planner imposes no treatment rationing. In other words, patient autonomy will induce an equilibrium outcome in which the range of treatment options accepted by patients increases with the queue length. However, the overall performance of the system under the competitive equilibrium is not socially optimal.

Next, we turn to investigate whether the ineffectiveness of treatment rationing is an artifact of the FCFS priority system. Our analysis demonstrates that, in fact, it is not. That treatment rationing when patients are homogeneous and autonomous is ineffective. Hence, the only meaningful management lever for the medical planner is the priority discipline. But FCFS is ineffective. Therefore, what is the priority discipline that maximizes social welfare when patients are autonomous? An examination of the FCFS system demonstrates that it is ineffective because future arrivals do not affect the patients already in the system, hence these patients do not consider the congestion externalities they impose when they decline a treatment option. Motivated by this observation, we then consider the extreme-opposite priority rule – Last-Come First-Served (LCFS) – and demonstrate that in this rule, patients internalize the externalities of their own decisions, and system performance achieves the socially optimal ideal.
This last finding brings to the forefront the equity-efficiency trade-off that underlies any medical waiting system. More importantly, our results demonstrate that patient autonomy exacerbates the trade-off. Specifically, it is a fundamental premise in medicine that FCFS is fair, and in that respect, LCFS can be blatantly unfair. It is therefore, natural, to consider a priority rule that is not as inefficient as FCFS but not as unfair as LCFS. In order to do that, we introduce a family of prioritization schemes, with both first-come first-serve and last-come first-serve belonging to that family, and examine the effect of these schemes. The main finding formalizes the intuition that in systems judged fair according to the FCFS criterion, patients internalize only a small fraction of the externalities caused by their autonomy, but in systems that are unfair according to the same criterion, patients internalize most of the externalities. This casts serious doubts on the validity of the premise that FCFS is a gold-standard for fairness. In a system with patient autonomy, one may argue that FCFS is the most “unfair” of all policies since the externalities of patient autonomy are borne by everyone except the person exercising this autonomy.

**Literature review.** Our work extends the extensive literature on individual versus social optimization in queueing systems. In our review of this research, we will use standard queueing-theoretic vocabulary. Naor (1969) was the first to analyze the effect of customer autonomy in queueing systems and he suggested that an admission toll is necessary to induce the social optimum. Yechiali (1971) considers the perspective of a profit-maximizing firm, and Lippman and Stidham (1977) and Stidham (1978) study the structural properties of the optimal congestion toll. The main message from these early papers is that when left to their own, customers join a queue without considering the waiting cost that is imposed on future customers, and Mendelson (1985) embeds the queueing model into an economic framework that directly considers the effect of such externalities. The incentive problems become more complex when there are multiple customer types because customers now have the added incentive to misrepresent themselves to obtain a better service level. Mendelson and Whang (1990) derive incentive-compatible priority pricing policies that simultaneously induce truth-telling and socially optimal behavior, and Van Mieghem (2000) combines this with dynamic scheduling policies. Another instance of similar incentive problems arises in the context of multi-server systems: arriving customers choose a server to join, and this
choice in general does not optimize the overall performance of the system. This problem is studied in Bell and Stidham (1983) and Lee and Cohen (1985). In relation to all these papers, our model exhibits two significant differences. Unlike previous work where the decisions of individuals on whether to join the queue, what priority level to purchase, and which server to choose are all made at the time of arrival, in our work the relevant decisions are made at the time of departure. That is, while customer autonomy induce state-dependent arrival rates in previous models, in our model we are faced with state-dependent service rates. Furthermore, in our work pricing is not a viable option.

In the health economics literature, most work on waiting lists is based on economic models that use cost-benefit analysis to quantify a socially optimal waiting list configuration; for a comprehensive survey see Cullis et al (2000). Incentive considerations in waiting lists have been attended to, but they focus mainly on the attitudes of physicians. Among others, Yates (1995) expresses concern over the possibility that the pursuit of private practice by NHS consultants may create a conflict of interest, and Weinstein (2001) contemplates the dual role of physicians as gatekeepers. The incentives of hospital management have also been studied, such as in Feldman and Lobo (1997), although to a lesser extent. The papers by Goddard, Malek and Tavakoli (1995) and Iversen (1997) examine patient preferences but differ from the current study in that they consider patients’ choices between seeking treatment from the waiting list or the private sector, whereas we are concerned with the behavior of patients while on the waiting list. Furthermore, the majority of papers in this area focus on static equilibrium models. Some exceptions include Goddard and Tavakoli (1994), who present a queueing analysis of the impact of various prioritization regimes, and Van Ackere and Smith (1997, 1999), who model the National Health Services waiting lists using system dynamics methodology. These papers, however, either assume that patients never decline treatment, or model such decisions exogenously. Therefore, our dynamic analysis of patient autonomy appears novel and hopefully sheds light on some trade-offs that have not been previously studied.

Relation to Other Waiting Systems. While our study is mainly motivated by medical waiting systems, the basic model has other important applications. Therefore, it is expected that the analysis and results will provide insights into a broader class of problems.
In particular, our model is highly relevant to managers in service industries concerned with customer satisfaction. In these systems, service is typically provided by many different servers with personalized attributes. It is conceivable that customers may choose to remain in line to wait for their favorite sales agent, bank teller, or hairdresser; even restaurant patrons sometimes choose to wait longer for a better or quieter table. It is therefore crucial to know what customers are willing to accept so that their needs can better be satisfied.

Another important application of our model is in the management of public housing programs where registrants may refuse to move into a public-housing project preferring to wait for a more attractive accommodation option. This kind of behavior leads to prolonged vacancy periods and a wastage of housing resources. To curtail such behavior, housing authorities have enforced harsh penalties, including removal from the waiting list and a ban from consideration for a fixed period of time. Kaplan in a series of four papers (1986, 1987a, 1987b, 1988) examines the queueing aspects of tenant assignment policies for public housing, and in (1987b) he also incorporates tenant choice as an exogenous acceptance probability. By internalizing customer choice, our model can improve our understanding of the underlying behavior and help formulate solutions that go beyond harsh penalties.

The remainder of the paper is organized as follows. A description of the model is presented in Section 2. Section 3 analyzes the socially optimal outcome when patients are not allowed to refuse treatment. The next section considers the competitive equilibrium that emerges under the FCFS priority discipline. A comparison of the socially optimal outcome to the outcomes from the competitive equilibrium verifies a welfare loss, whose source is examined in detail in Section 5. The subsequent two sections verify that prioritization is a potent policy instrument. Specifically, Section 6 shows that the socially efficient outcome can be recovered by using LCFS, and Section 7 discusses a family of prioritization schemes that capture the essence of the efficiency-equity trade-off underlying the selection of different priority schemes. Section 8 presents a numerical example that investigates the magnitude of the welfare losses under FCFS and the model parameters that most contribute to these losses. Because our basic model is highly simplified, we explore the implications of several extensions in Section 9. Concluding remarks are presented in Section 10. All proofs are relegated to the Appendix.
2 Problem Formulation

In this section, we introduce the queueing model for the medical waiting list. The model assumes that patients arrive according to a time-homogeneous Poisson process with rate \( \lambda \), and treatment opportunities arrive according to an independent Poisson process with rate \( \mu \). Patients can depart from the waiting list either when they receive and accept a treatment opportunity, or they can renege after an exponentially distributed amount of time with rate \( \gamma \). The reneging process is independent from both patient arrivals and treatment arrivals. For brevity of notation, it is convenient to normalize the treatment arrival rate \( \mu \equiv 1 \).

The reward structure for the model assumes that patients are assigned a quality of life score that depends on whether they are waiting, they have received a treatment, or they have reneged. The quality of life score reflects the desirability of each of the three states (waiting, post-treatment, post-reneging) and patient preferences are homogeneous in the sense that all patients have the same quality of life score in each state. Patients waiting receive a continuous payoff at a rate \( h \) per unit time. Patients that renege receive an instantaneous reward \( d \) which reflects the total expected discounted quality adjusted life years post-reneging. In addition, patients receive a payoff upon treatment. To reflect treatment variability, we assume that there exists an abstract measure of “treatment quality” captured by a continuous random variable \( X \) which takes values in \((\underline{x}, \overline{x})\) and has probability density \( f \). When a treatment opportunity arrives, its quality \( X = x \) is realized and publicly observed, and this reflects the post-treatment total expected discounted quality adjusted life years for the patient receiving the treatment. All rewards will be continuously discounted at rate \( \beta \). For future reference, it is also convenient to define the cumulative probability distribution \( \bar{F}(x) = \int_{x}^{\overline{x}} f(x)dx \) and also the function \( g(x) = \int_{\underline{x}}^{x} f(x)dx / \bar{F}(x) \) which gives the expected conditional reward from treatment that exceeds a threshold \( x \).

Patients have the option to decline a treatment offer in anticipation of a potentially better future offer, and their objective is to exercise their “decline” option carefully in order to maximize their total quality adjusted life expectancy. A policy for a patient specifies the range of treatment options that are acceptable by the patient at each point in time. The policy can be stationary or non-stationary. It can also be history-dependent. Similarly, the medical planner (referred to as “she”) must decide how to allocate the treatment options
that become available and her objective is to maximize the total quality adjusted expected life years of all patients. A policy for the medical planner specifies at each point in time the range of treatment options that will be assigned to patients waiting and the rank of all patients in the waiting list. A treatment option is assigned to the top ranked candidate. If she or he declines it, then the rankings may or may not change, and the option will be offered to the next ranked candidate. The process will be repeated until the treatment option is either accepted by someone on the waiting list or it is declined by everyone. The policy can either be stationary or non-stationary, and can also be history-dependent. It is also assumed that each patient and the medical planner have perfect information about the queue length and the quality of treatment. Also, all model primitives are known to all parties.

Before we proceed further, it is worthwhile to pause and reflect on the main assumptions made in the model. First, the current specification is very general and does not require additional assumptions on the parameters. There are no restrictions on the signs of \( h \) or \( d \), although finiteness on the bounds \( \underline{\tau}, \overline{\tau} \) is required. For ease of exposition, we will develop our analysis assuming that the quality random variable \( X \) has a continuous density \( f \), but this assumption is not necessary; for example, \( X \) could be a discrete random variable. Furthermore, the analysis can be carried over to the undiscounted case by taking the limit \( \beta \to 0 \). Second, the model includes several restrictive assumptions. While a detailed discussion about their implications will be postponed until after the analysis of the model is complete, we find it convenient to briefly summarize them now in order to increase the reader's awareness. The most restrictive assumptions are that patients are homogeneous, that the treatment reward does not depend on the time since enrollment, that the quality of the treatment option is perfectly observable when the option becomes available, and that the state of the system as reflected by the queue-length size is also perfectly observable. It is natural to determine whether the main conclusions from our analysis will change if some of these assumptions are relaxed, and this will be discussed in section 9.
3 Socially Optimal Outcome

We now consider the problem of the medical planner when patients cannot reject a treatment offer. The planner’s policy specifies a “treatment acceptance region” for each possible queue length, and for tractability we will assume that the region is characterized by a threshold such that acceptable treatments take values above the threshold. Let $b(n)$ denote the acceptance threshold when the queue length is $n$, and let $V(n)$ denote the medical planner’s optimal total expected discounted reward. Bellman’s equation of optimality states that:

$$V(0) = \frac{\lambda}{\beta + \lambda} V(1)$$

(1)

$$V(n) = \sup_{b(n) \in [x, \infty]} \frac{1}{\beta + 1 + \lambda + \gamma n} [hn + F(b(n))(V(n - 1) + g(b(n))) + (1 - F(b(n)))V(n) + \lambda V(n + 1) + \gamma n(V(n - 1) + d)].$$

(2)

To interpret (2), notice that the total waiting payoff rate for all patients is $hn$ when there are $n$ patients on the waiting list. This is earned continuously until the next transition which occurs with rate $1 + \lambda + \gamma n$. Then, there are four possible transitions: a) With probability $\frac{F(b(n))}{1 + \lambda + \gamma n}$, a treatment option that exceeds the threshold $b(n)$ occurs, in which case the treated patient receives an instantaneous payoff of $g(b(n))$, and the waiting list decreases by one; b) With probability $\frac{1 - F(b(n))}{1 + \lambda + \gamma n}$, a treatment option that is below the decision threshold occurs, but this does not affect the state of the system since the option is discarded; c) A new arrival occurs with probability $\frac{\lambda}{1 + \lambda + \gamma n}$ and this increases the queue length by one; and d) A patient reneges with probability $\frac{\gamma n}{1 + \lambda + \gamma n}$, in which case the queue length decreases by one and an instantaneous payoff of $d$ is accrued. Equation (1) is a boundary condition which states that there is no waiting payoff when the system is empty and that the only possible transition is a new patient arrival occurring at a rate $\lambda$.

The optimality equation has two important corollaries. First, it can be shown that the optimal decision thresholds $b^*(n)$ are related to the optimal value function as follows (assuming that the constraint $b^*(n) \geq x$ is not binding):

$$b^*(n) = V(n) - V(n - 1).$$

(3)

This expression can be derived from the first-order conditions for the maximization problem in the right-hand side of (2) and it is valid because the function in the right-hand side
of (2) is concave in \( b(n) \)– it has a non-positive second derivative. More intuitively, this expression follows because the medical planner makes each threshold decision by weighing two alternatives: (i) approve treatment and earn a payoff of at least \( b(n) + V(n - 1) \), or (ii) deny treatment and maintain the continuation payoff \( V(n) \). Therefore, the expression in (3) states that the medical planner is indifferent between these two choices on the margin.

The second corollary is that the decision thresholds are non-increasing in the queue length. That is, the medical planner is selective in the choice of treatment options when the queue length is small, but as the queue length increases, less-effective treatments also become acceptable.

**Proposition 1** The socially optimal decision thresholds \( \{b^*(n)\} \) are non-increasing in \( n \).

The result is proven in the appendix using an argument presented in Bertsekas (1995) and extended by George and Harrison (1999).

### 4 Competitive Equilibrium under FCFS

We now extend the analysis to explore the competitive equilibrium that emerges when patients retain their autonomy and the medical planner uses the FCFS priority discipline. To pursue this task we must provide a precise definition of the medical planner’s and of each patient’s strategy, introduce the appropriate equilibrium concept and finally provide an algorithm that derives the equilibrium. Throughout this section, we will focus on Markovian strategies for all parties.

Consider first the medical planner. Her strategy consists of the FCFS priority rule and a queue-length-dependent rationing rule \( b = \{b(n) : n = 1, ..., \infty\} \). Under this rule, when the queue length is \( n \), only treatment options with a value no less than \( b(n) \) will be assigned to waiting patients, and patients are prioritized according to their arrival time. When a patient declines a treatment option, then that treatment is offered to the next candidate on the waiting list.

Next, consider the patients on the waiting list. Rather than characterize the strategy of each patient, we define a strategy profile that characterizes the behavior of all patients. We
will focus on Markovian strategies in which each patient’s decision on the range of acceptable treatments depends on the current queue length and the patient’s position on the waiting list. Hence, a strategy profile is characterized by a set of thresholds \( \{a_k(n) : n \geq 1, n \geq k \geq 1\} \) that admit the following interpretation: When the queue length is \( n \), the patient on position \( k \) will only accept treatment options when their value is no less than the decision threshold \( a_k(n) \); otherwise the patient retain his position in line and pass on the treatment opportunity to the next patient. For brevity of notation, we let \( a = \{a_k(n) : n \geq 1, n \geq k \geq 1\} \). In addition, given the medical planner’s threshold strategy \( b \), it follows that a strategy profile must be such that

\[
a_1(n) \geq a_2(n) \geq ... \geq a_n(n) \geq b(n); \tag{4}
\]

that is the threshold for the \( k \)th position cannot be less than the threshold for the \( k+1 \)st position, and all thresholds are no less than the medical planner’s threshold. This condition follows from the FCFS priority rule which implies that violations of this condition are not implementable: the decision thresholds for patients that are higher on the priority list determine the feasible thresholds for all patients behind them, and the medical planner’s decision threshold places a lower bound on everyone’s thresholds.

In this setting, the different patients are involved in a multi-stage infinite horizon game and the relevant equilibrium concept is that of subgame perfection. In this concept, a strategy profile is subgame perfect if patients cannot gain by unilateral one-stage deviations from the equilibrium strategy. That is, there is no state, defined by the pair \((n, k)\) of the queue length and patient position, where the \( k \)th patient may gain by deviating from the actions prescribed by the strategy profile at this state but complying to the strategy thereafter.

We will now provide an algorithm that identifies a subgame perfect strategy profile for this game. This involves several steps. First, let \( V_k^a(n) \) denote the total expected discounted payoff for the patient in position \( k \) when the queue length is \( n \) and all patients comply with
strategy profile \( a \). Next, we define the operator \( T^a \), which must be such that \( V^a = T^a V^a \):

\[
T^a V_1^a(1) = \frac{1}{\beta + 1 + \lambda + \gamma} [h + \mathcal{F}(a_1(1))g(a_1(1)) + (1 - \mathcal{F}(a_1(1)))V_1^a(1) \\
+ \lambda V_1^a(2) + \gamma d] \\
T^a V_1^a(n) = \frac{1}{\beta + 1 + \lambda + \gamma n} [h + \mathcal{F}(a_1(n))g(a_1(n)) \\
+ (\mathcal{F}(a_n(n)) - \mathcal{F}(a_1(n)))V_1^a(n - 1) + (1 - \mathcal{F}(a_n(n)))V_1^a(n) \\
+ \lambda V_1^a(n + 1) + \gamma d + (n - 1)\gamma V_1^a(n - 1)] \\
T^a V_n^a(n) = \frac{1}{\beta + 1 + \lambda + \gamma n} [h + \mathcal{F}(a_{n-1}(n))V_n^a(n - 1) \\
+ \mathcal{F}(a_n(n))g(a_{n-1}(n)) - \mathcal{F}(a_{n-1}(n))g(a_n(n)) + (1 - \mathcal{F}(a_n(n)))V_n^a(n) \\
+ \lambda V_n^a(n + 1) + \gamma d + (n - 1)\gamma V_n^a(n - 1)] \\
T^a V_{k-1}^a(n) = \frac{1}{\beta + 1 + \lambda + \gamma n} [h + \mathcal{F}(a_{k-1}(n))V_{k-1}^a(n - 1) \\
+ \mathcal{F}(a_k(n))g(a_{k-1}(n)) - \mathcal{F}(a_{k-1}(n))g(a_k(n)) \\
+ (\mathcal{F}(a_n(n)) - \mathcal{F}(a_k(n)))V_{k-1}^a(n - 1) + (1 - \mathcal{F}(a_n(n)))V_{k-1}^a(n) \\
+ \lambda V_{k-1}^a(n + 1) + (k - 1)\gamma V_{k-1}^a(n - 1) + \gamma d + (n - k)\gamma V_{k-1}^a(n - 1)] \\ (8)
\]

While this definition of the continuation payoff operator is daunting, it is very similar to the standard definition for the dynamic programming operator used in the single party dynamic decision making problem (1)-(2). The differences stem from the multiparty nature of the problem at hand, and we will interpret them by examining (8) which gives the most general case; the interpretation for equations (5)-(7) follows a similar line of thinking.

First, the definition implies that the operator \( T^a V^a \) gives the continuation payoffs for a two-stage problem in which the actions taken in the first stage are the ones prescribed by the strategy profile \( a \) and with a terminal payoff \( V^a \) in the second stage. Now, to explore the individual components of (8) notice that \( \frac{h}{\beta + 1 + \lambda + \gamma n} \) is the expected waiting payoff until the next transition, which occurs at a rate \( 1 + \lambda + \gamma n \). Then, there are three possible transitions: treatment opportunity arrival (uniformized to rate 1), patient arrival (rate \( \lambda \)), and reneging (rate \( \gamma n \)). We shall systematically consider each possible type of transition, its probability of occurrence, and its associated continuation payoff. These are the possible cases:

1. Treatment Opportunity Arrivals: When a treatment opportunity arrives, there are four
possibilities: a) Its value exceeds \( a_{k-1}(n) \) which implies that it will be accepted by one of the first \( k-1 \) patients leading into a reduction in the queue length and a reduction in the \( k \)th patient’s position; b) Its value will be between \( a_{k-1}(n) \) and \( a_k(n) \) implying that it will be accepted by the \( k \)th candidate, who will receive the reward and depart; c) Its value will exceed \( a_{n}(n) \) but will be less than \( a_k(n) \) implying that it will be accepted by one of the candidates between positions \( k \) and \( n \) leading into a reduction in the queue length; d) Its value will be less than the threshold \( a_{n}(n) \) implying that the treatment opportunity is discarded and the state of the system is unchanged.

2. New Patient Arrivals: This transition occurs with probability \( \frac{\lambda}{1+\lambda+\gamma n} \), it leaves the position of the \( k \)th patient unchanged, and it increases the queue length by one.

3. Patient Reneging: This occurs with rate \( n\gamma \) and there are three distinct possibilities:
   a) One of the \((k-1)\) patients in front of the \( k \)th patient reneges advancing the position of the \( k \)th patient and decreasing the queue length by one (this transition occurs with probability \( \frac{(k-1)\gamma}{1+\lambda+\gamma n} \));
   b) The \( k \)th patient reneges with probability \( \frac{\gamma}{1+\lambda+\gamma n} \) and leaves the system with instantaneous payoff \( d \);
   c) One of the \((n-k)\) patients behind the \( k \)th patient reneges decreasing the queue length by one but leaving the \( k \)th patient’s position unchanged (this transition occurs with probability \( \frac{(n-k)\gamma}{1+\lambda+\gamma n} \)).

Having completed the derivation of the operator \( T^a \), we are in a position to formally define the equilibrium concept as follows: A strategy profile \( a^F \) is a subgame perfect equilibrium if it attains the following supremum (where the supremum is taken over one component of the strategy profile at a time):

\[
V^F = \sup_{\{a_k(n): a_1(n) \geq a_2(n) \geq \ldots \geq a_n(n) \geq b(n)\}} T^a V^F; \tag{9}
\]

The fixed point \( V^F \) is the continuation payoff for the equilibrium strategy \( a^F \); the superscript \( F \) here represents an abuse of notation and indicates that the priority rule is First-Come First-Served. The shorthand notation (9) indicates that \( a^F \) is a subgame perfect equilibrium strategy if no patient can gain by unilateral one-stage deviations from the actions prescribed by the equilibrium strategy. Furthermore, it can be shown that (9) has a unique solution for
the continuation payoff since the operator $T^a$ is a contraction mapping, but the equilibrium strategy is not necessarily unique.

Having completed the characterization of the equilibrium strategy profile, we are now in a position to derive a simple expression relating the strategy profile to the continuation payoffs. Specifically, as in the socially optimal case, it is straightforward to confirm that the first order optimality conditions are necessary and sufficient (ignoring momentarily the boundary constraints $a_k(n) \geq a_{k+1}(n) \geq b(n)$), which implies that the following conditions hold:

\begin{align}
a_1^F(1) &= V_1^F(1) \\
da_k^F(n) &= V_k^F(n - 1) \\
da_n^F(n) &= V_n^F(n).
\end{align}

Intuitively, these conditions state that a patient is indifferent between accepting the marginal treatment $a_k^F(n)$ and passing it along to the next patient on the waiting list. If he passes it to the next patient, then that patient will accept it, and hence the queue length decreases by one leaving him with a continuation payoff $V_k^F(n - 1)$. Similar interpretations are valid in the boundary cases where the patient is either the only one waiting, or the last one waiting. In those cases, the patient's position and queue length remain unchanged if the marginal treatment is refused.

Having characterized the equilibrium strategy profile that emerges as a response to the medical planner’s treatment allocation policy, we can now take a step back and ask the following question: Given our prediction for the equilibrium strategy profile, what is the optimal treatment rationing strategy $b$ that would maximize the medical planner’s objective?

The following proposition, proven in the appendix, provides the answer.

**Proposition 2** It is optimal to impose no treatment rationing by choosing $b(n) \equiv \pi$ for all $n$.

The main observation employed in the proof is that the medical planner’s decision thresholds $\{b(n)\}$ appear in the equilibrium characterization (9) only as a constraint on the feasible decision thresholds for the patients. Therefore, by eliminating these bounds one
increases the total expected quality adjusted life years for all patients, hence increasing the medical planner’s overall objective.

Before we interpret this finding, a caveat is in place. In our analysis, we have restricted attention to simple threshold policies where the acceptable treatment options for the medical planner is the interval \([b(n), \bar{x}]\) when the queue length is \(n\). While it is conceivable that the medical planner may wish to allow a more general domain, it can still be shown that the optimal policy is to exercise no control for exactly the same reasons. Hence, our restriction has been without loss of generality.

Let us now interpret this finding. First, it states that the medical planner makes the waiting system less efficient by implementing explicit treatment controls, because such controls merely restrict the patient’s strategy space and hence are ineffective. Interpreted more negatively, this result states that treatment control is not a useful policy instrument when patients are autonomous. Or, more broadly, in queueing systems with homogeneous customers, service rate control is ineffective.

From a computational standpoint, this result simplifies the derivation of the equilibrium strategy profile. Specifically, when the medical planner imposes no control on the range of feasible treatment options, patients no longer have to consider waiting list sizes in their decision problem. The only relevant information is their position on the waiting list. This is in stark contrast to the case where the medical planner imposes control because then queue length matters due to its effect on the range of treatment options available to the patients. This implies that \(V_k^F(n)\) is independent of the queue length and depends only on the patient’s position \(k\); thus we let \(V_k^F(n) = V^F(k)\) and \(a_k^F(n) = a^F(k)\). With this simplification, it follows that the equilibrium strategy profile is derived as follows:

\[
V^F(1) = \sup_{a(1) \in [x, \bar{x}]} \frac{1}{\beta + 1 + \gamma} [h + \bar{F}(a(1))g(a(1)) + (1 - \bar{F}(a(1)))V^F(1) + \gamma d]
\]

\[
V^F(k) = \sup_{a(k) \in [x, a(k-1)]} \frac{1}{\beta + 1 + \gamma k} [h + \bar{F}(a(k-1))V^F(k - 1) + \bar{F}(a(k))g(a(k))
- \bar{F}(a(k-1))g(a(k-1)) + (1 - \bar{F}(a(k)))V^F(k) + \gamma d + (k - 1)\gamma V^F(k - 1)]
\]

This simplified characterization of the equilibrium strategy profile also reveals the main shortcoming of FCFS. First, note that absent any explicit control by the planner, the
system with autonomous patients achieves an implicit decision threshold such that only treatments with quality greater than $a^F(n)$ are accepted when the queue length is $n$. However, these equilibrium thresholds obtained by (13)-(14) will deviate from the socially optimal thresholds $b^*(n)$ because in choosing $a^F(n)$ patients ignore future arrivals. A more detailed examination of this finding will be pursued in the next section.

5 Welfare Loss from Patient Autonomy

We now wish to investigate in more detail the welfare losses caused by patient autonomy. To do that, we will compare the socially optimal decision thresholds $b^*(n)$ to the competitive equilibrium thresholds $a^F(n)$ identified in Section 4. It will be shown that the latter can be obtained by solving a dynamic programming recursion analogous to the ones solved in socially optimal case (1)-(2), but with the patient arrival rate set at $\lambda = 0$. Hence, the competitive thresholds are inefficient because they are derived by ignoring future arrivals. Specifically, the argument proceeds by considering the aggregate value function $\bar{V}^F(n) = \sum_{k=1}^{n} V^F(k)$. Then it can be shown that $\bar{V}^F(n)$ and $a^F(n)$ are obtained by solving the dynamic programming recursion (1)-(2) but with the exogenous arrival rate set at $\lambda = 0$. This statement is made precise in the following proposition which is proven in the Appendix.

**Proposition 3** Under FCFS, the solution $\{a^F(n)\}$ to the optimality equations (13)-(14) also solves (1)-(2) but with the arrival rate $\lambda = 0$.

The proof demonstrates that the decision threshold $a^F(n)$ maximizes the value function of the $n$-th patient and the sum of the continuation payoffs of all preceding patients. This is not surprising since these preceding patients have a “first pick” on treatment options, so the $n$-th patient, who has no control over their choices, will perceive their utility as a constant. Thus, by maximizing the total utility of all preceding patients, the $n$-th patient also maximizes his or her own payoff.

This result suggests that if the social planner is faced with a hypothetical system with an arrival rate of zero, the optimal controls he will choose for this hypothetical system will coincide with the competitive equilibrium under FCFS. In other words, the welfare loss that
is observed arises because the self-serving behavior of patients distorts the system into one where control decisions are made by ignoring future arrivals. The intuition is that under FCFS, new patients do not affect existing patients, so they do not factor into the calculation of the optimal decision thresholds for the existing patients. As a consequence, it can also be shown that the competitive thresholds are higher than the socially optimal ones:

$$a^E(n) \geq b^*(n),$$

(15)

implying that the competitive outcome is inefficient because patients are too stringent, and thus generate congestion externalities. A formal proof for this result will be presented in section 7 where a general class of priority rules and their corresponding equilibria will be analyzed.

In summary, our analysis has culminated into the somewhat dismal message that treatment rationing is an ineffective way to control patient behavior, and that the system is inefficient under the commonly used FCFS rule. However, we shall show in the next section that social efficiency can be achieved if the priority rule is LCFS.

6 The Role of Prioritization

The analysis for the LCFS discipline follows the steps developed in section 4 with the exception that each new patient now arrives at the top of the line, shifting back the position of all other patients by one. Imitating the analysis of section 4, we can easily show that the medical planner will never restrict the range of treatment options offered to the patients, and hence, the value function for a patient in position $n$ does not depend on the queue length. If we now let $V^L(n)$ denote the value function for a patient in position $n$ and $a^L(n)$ denote the optimal treatment acceptance thresholds, then the dynamic programming recursion for
the competitive equilibrium is as follows:

\[ V^L(1) = \sup_{a(1) \in [x, \tilde{x}]} \frac{1}{\beta + 1 + \lambda + \gamma} \left[ h + \tilde{F}(a(1))g(a(1)) + (1 - \tilde{F}(a(1)))V^L(1) \right. \]
\[ \left. + \lambda V^L(2) + \gamma d \right] \]  
\[ (16) \]

\[ V^L(n) = \sup_{a(n) \in [x, a(n-1)]} \frac{1}{\beta + 1 + \lambda + \gamma n} \left[ h + \tilde{F}(a(n-1))V^L(n-1) + \tilde{F}(a(n))g(a(n)) \right. \]
\[ \left. - \tilde{F}(a(n-1))g(a(n-1)) + (1 - \tilde{F}(a(n)))V^L(n) \right. \]
\[ \left. + \lambda V^L(n+1) + \gamma d + (n-1)\gamma V^L(n-1) \right]. \]  
\[ (17) \]

Unlike the FCFS systems, in the LCFS system arrivals are taken into account because they cause each patient’s position to increase by one. One can proceed by considering the aggregate value functions to compare the competitive acceptance thresholds under LCFS to the socially optimal thresholds. The following result shows that the aggregate value function is obtained by solving the dynamic programming recursion (1)-(2), and hence, with the LCFS priority rule, the medical waiting system with autonomous patients is socially efficient.

**Proposition 4** Under LCFS, the optimal strategy for the medical planner is to exercise no treatment control. The competitive equilibrium that emerges, \{a^L(n)\}, is socially optimal.

We shall defer the proof of this proposition until a more general version of this result will be presented in section 7.

The key observation when comparing the results under FCFS and LCFS rules is that the welfare loss under consideration stems from the potential negative externality that are imposed on future patients when treatment opportunities are refused by existing patients. It is important to identify the victims of this externality: taking a system-level perspective, observe that the effect of one patient refusing treatment is inconsequential as long as this opportunity is accepted by another patient further down along the waiting list; the difference arises only when a treatment opportunity is declined by every patient, in which case the current pool of patients has collectively imposed a negative externality on future patient arrivals who will have to face a longer waiting list. Among all current patients who impose this externality, the only patients who would internalize it are the ones who will be placed behind this new patient when he or she arrives. Therefore, with LCFS prioritization this
externality becomes completely internalized because the new patient will arrive in front of all existing patients. On the other extreme, the effect of this externality is greatest under FCFS prioritization because new patients are placed behind existing patients that generate these externalities. This explanation does not imply that the system will be efficient as long as no treatment opportunities are being wasted. When waiting lists are long, as is often the case in practice, it is rare for a medical resource to be discarded, but these externalities still influence system transition rates and result in longer queues and longer waiting times. Therefore, even if treatment options are discarded infrequently, they can exacerbate system congestion and aggravate patient waiting time.

Our result provides a rigorous proof for the statement in Hassin’s (1985) study on the social optimality of the LCFS rules. However, the result should be treated with caution. As noted in Hassin (1985), there is a strategic difficulty associated with LCFS prioritization. Without any form of monitoring, any person in line has the motivation to balk and re-enter the system at the top of the line. Although we shall assume that this is not permitted, we also acknowledge the massive administrative costs involved with preventing such behavior, which could also partly explain why LCFS systems are rarely observed in practice.

On the other hand, the implication that FCFS prioritization is, in some sense, the least efficient appears incompatible with the observation that it is very commonly used in practice. In fact, apart from minor provisions made for exceptional cases, it is the primary prioritization scheme being used. While it is recognized in Larson (1987) that FCFS prioritization, being a symbol of justice and equity, enjoys many advantages that go beyond economic efficiency, our model does not admit a quantitative consideration of this dimension. In fact, our result may suggest that FCFS is fundamentally unjust since the externalities generated by a patient’s decision to decline a treatment are not internalized by that patient, but rather by all future patients.

7 The Efficiency-Equity Trade-off

Our analysis so far has demonstrated that the LCFS priority rule achieves a socially efficient competitive equilibrium, while the most common FCFS rule appears ineffective because the
patients who decline a treatment offer do not internalize the externalities they impose on future arrivals. This observation brings to the forefront the trade-off between efficiency and equity: The priority rule that maximizes system efficiency is the one the deviates the most from FCFS, the acceptable gold-standard for equity.

In order to quantify this trade-off, we now consider a continuum of priority rules which we call randomized absolute priority rules. In these rules, new patients will either receive absolute priority and skip to the head of the line, or join the end of the line. Each new patient is granted absolute priority independently with probability \( p \in [0, 1] \); we let \( \{a^p(n)\} \) denote the equilibrium decision thresholds under this priority system. The case of \( p = 0 \) corresponds to FCFS and \( p = 1 \) corresponds to LCFS. Although these prioritization schemes are probabilistic hybrids of the FCFS and LCFS rules and thus natural candidates for analysis, the more important reason for choosing them is that some form of absolute priority is often observed in practice. For example, hospital waiting lists give absolute priority to emergency cases, and kidney transplant waiting lists give absolute priority to candidates possessing perfect protein matches with the donor organ. The violation of FCFS occurs at the time of patient arrival in one case and at the time of patient departure in the other, but both cases exhibit absolute priorities that can be approximated by our randomized absolute priority rules. This simple class of prioritization schemes can even be used to approximate the behavior of priority queues in service systems when the size of the high-priority “queue” is rarely greater than one, which occurs with either a high service rate or a low arrival rate for the high-priority class.

The reader could almost predict our next result, which states that the effective arrival rate is \( \lambda p \) under the randomized absolute priority rule with parameter \( p \). The proof, presented in the appendix, relies on the aggregation argument that we have described in section 5.

**Proposition 5** Under randomized absolute priority with parameter \( p \), the competitive equilibrium thresholds \( \{a^p(n)\} \) can be obtained by solving the optimality equations (1)-(2) with the arrival rate set at \( \lambda p \).

This proposition presents a continuum of cases covering both FCFS and LCFS prioritization, and confirms the antithetical relationship between absolute priority and externalities. Absolute priority embodies nuances of social injustice, while negative externalities,
manifested through distortions in arrival rates, lead to economic inefficiency. The relationship between the absolute priority parameter $p$ and the effective arrival rate $\lambda p$ can thus be interpreted as a quantitative representation of the efficiency-equity trade-off, since absolute priorities provide a mechanism to minimize the impact of the negative externalities.

The effect of the priority parameter $p$ on controlling the externalities can also be confirmed in the following result which states the impact of the parameter on the equilibrium thresholds.

**Proposition 6** Let $0 \leq p \leq p' \leq 1$. Then, for each queue length $n$,

$$a^p(n) \geq a^{p'}(n).$$

(18)

This proposition states that patients become less stringent in their decision rule as the priority parameter increases, and consequently as the threat of a reduction in their priority following a treatment decline becomes more severe. Therefore, the social planner is able to mitigate the externality effects with randomized absolute priority rules. Such regimes can vary in intensity according to the parameter $p$, and more importantly, can be justified in many ways. On a broader level, our interpretations also suggest that these externality problems can be kept under control using a more general class of preemptive regimes that go beyond randomized absolute priority rules - regimes in which patients that decline a treatment offer would expect a decrease in their priority position.

So far, we have approached the subject of waiting list management by considering two policy instruments: treatment rationing and prioritization. On one hand, we find that treatment rationing is ineffective because the best decision for the medical planner is not to impose any control at all, but on the other hand, we find that an optimal prioritization scheme could coordinate the system to achieve social efficiency. These conclusions are reversed in standard queueing models with homogeneous customers in which prioritization is irrelevant, but treatment rationing (commonly referred to as service rate control) is paramount. This stark contrast reveals that the consideration of patient autonomy can lead to very different policy recommendations, and explains why prioritization schemes are used in practice. Prioritization is a fine-level policy instrument that affects every individual at any
time and can serve as an important lever to balance the subtle trade-off between efficiency and equity.

8 Numerical Study

In this section, we present results from a numerical study that has two main objectives. The first one is to illustrate that patient autonomy can significantly degrade the performance of medical waiting lists as measured by average waiting time and expected patient reward, and to identify the key system parameters that either exacerbate or alleviate such performance degradation. The second objective is to study the effect of absolute priorities on system performance. In particular, it will be demonstrated that a system where a small fraction of the patients are granted absolute priorities, while the rests are prioritized according to FCFS can recover most of the losses caused by the FCFS system.

How significant is the welfare loss? To address this question consider a system where patients are autonomous and with the following parameters: arrival and service rates are set at $\lambda = \mu = 100$, the individual reneging rate is $\gamma = 1$, the payoff after reneging is $d = 0$, the payoff rate while waiting is $h = 60$, and the random treatment payoff $X$ is uniformly distributed between 50 and 90. These payoff figures can be interpreted as quality-of-life scores, normalized between 0 and 100, and will be discounted at a continuous rate of $\beta = 1$. Notice that with our assumed payoff structure, the patient’s expected payoff from waiting indefinitely is $\frac{h}{\beta+\gamma} = 30$, hence even the worst treatment option is attractive.

The performance of this system is evaluated under both FCFS and LCFS. Our analysis allows us to compute system transition rates, stationary queue-length distributions and value functions - these quantities can in turn be used to compute the following performance metrics of interests: average waiting time, average queue length, total discounted reward expected by a randomly arriving patient. For both FCFS and LCFS, Figure 1 plots the stationary queue length distributions (thin lines) and the state-dependent capacity utilization as functions of the queue length $n$; capacity utilization in state $n$ indicates the fraction of treatment options accepted in that state. The bold lines show that the percentage of treatment options accepted under FCFS is far lower that the socially optimal levels achieved under LCFS. This
Figure 1: Performance Evaluation Under FCFS and LCFS

results into longer queue lengths and longer waiting times. Indeed, the queue length density curve for FCFS stochastically dominates the LCFS curve. Table 1 summarizes some key performance metrics. The results demonstrate that patient autonomy in the FCFS system doubles the mean queue length, waiting time until treatment and fractions of deaths, relative to the socially optimal LCFS. This massive disparity culminates into a 12% decrease in the expected discounted reward for each patient.

To examine the impact of system parameters on welfare losses, we obtain the same per-

<table>
<thead>
<tr>
<th></th>
<th>Mean Queue Length</th>
<th>Waiting Time Till Treatment</th>
<th>Fraction of Deaths</th>
<th>Expected Reward</th>
</tr>
</thead>
<tbody>
<tr>
<td>FCFS</td>
<td>20.50</td>
<td>0.22</td>
<td>0.21</td>
<td>57.62</td>
</tr>
<tr>
<td>LCFS (First Best)</td>
<td>8.46</td>
<td>0.09</td>
<td>0.08</td>
<td>65.72</td>
</tr>
<tr>
<td>FCFS/LCFS</td>
<td>2.43</td>
<td>2.56</td>
<td>2.42</td>
<td>0.88</td>
</tr>
</tbody>
</table>

Table 1: Welfare Loss in Terms of Performance Metrics
formance metrics under four additional scenarios representing an increased scale, increased treatment variability, increased treatment quality, and increased supply. The results are summarized in Table 2.

The following four key observations can be made.

1. When the system is scaled up by increasing arrival and service rates fifty-fold, the welfare loss of FCFS compared to LCFS increases from 12.3% to 16.7%. System scaling increases expected rewards only marginally under FCFS (from 57.62 to 57.93) but more substantially under LCFS (from 65.72 to 69.51). This suggests that the statistical economies of scale that are frequently present in non-autonomous queueing systems, cause an increase in welfare losses when the entities populating the system are autonomous.

2. By changing the reward distribution from $U[50, 90]$ to $U[40, 100]$, the variability in treatment options faced by patients increases, while the mean is preserved. This leads to an increased welfare loss because the option to wait becomes more valuable in an environment of high variability. The upside potential from waiting becomes more pronounced.

3. A uniform improvement in treatment quality by five units causes a decrease in welfare loss. However, despite this increase in treatment quality, the performance of the FCFS system fails to surpass the socially optimal performance of the baseline system. This implies that remedying the externality problems in the original system would have
larger welfare impact compared to an increase in average treatment quality.

4. In the same vein, increasing treatment supply by 40% leads into a welfare loss reduction compared to LCFS. However, the absolute improvement realized under FCFS is almost of the same order of magnitude as the welfare loss from FCFS in the baseline system. That is, the welfare gain caused by the adoption of the LCFS rule compared to FCFS is equivalent to a welfare gain that can be achieved by a 40% increase in supply of treatment.

These results suggest that large scale real-life medical waiting systems suffer from substantial welfare losses caused by patient autonomy. While most efforts to improve the performance of such systems focus on supply-side strategies, our analysis suggests that the effectiveness of supply-side interventions pales in comparison to that of the demand-based priority rules proposed here.

**How much absolute priority should be granted?** While the previous analysis suggests that the welfare gains from switching to LCFS compared to the common FCFS are substantial, there are several hurdles that would prevent the adoption of LCFS. In fact, most medical waiting lists adopt a modified FCFS rule where a fraction of newly arrived patients are granted absolute priority. Our analysis in section 7 suggests that such priority systems can recover some of the welfare losses since patients internalize some of the externalities caused by their decision to decline a treatment that is attractive from a system-wide perspective. In the remainder of this section, we focus on the relation between the fraction of patients receiving absolute priorities, and the portion of the social welfare attained compared to the socially optimal LCFS.

We start with the baseline scenario and investigate how the performance of the system (measured as the expected total discounted reward per patient) changes as the fraction $p$ of patients who receive absolute priority also changes. The solid line in Figure 2 summarizes our results, plotting the fraction of welfare loss recovered against the parameter $p$. As expected, the fraction of welfare loss recovered increases from 0 to 1 as we increase $p$ from 0 (FCFS) to 1 (LCFS). The crucial observation, however, is that this curve is concave, indicating that the marginal impact of the first few units of absolute priority is the greatest.
Next, we repeat the experiment and introduce an imbalance between supply and demand. We consider the following two cases ($\lambda = 200, \mu = 100$ and $\lambda = 500, \mu = 100$), and represent the results using dotted lines in Figure 2. With a supply shortage, our earlier observations become more pronounced. When demand is twice as much as supply, 90% of the welfare loss is recovered with $p = 0.42$, and the socially efficient outcome is attained with $p = 0.48$. When demand is five times greater than supply, 90% of the welfare loss is recovered with $p = 0.07$, and the socially efficient outcome is attained with $p = 0.09$. The general observation is that in situations of supply imbalances, small levels of absolute priority are highly effective and diminish the negative externalities caused by patient autonomy.
9 Discussion of Main Model Assumptions

Our analysis has relied on a highly stylized analytically tractable model of a medical waiting system. While it is virtually impossible to develop a tractable model that captures every single aspect of reality, it is nevertheless important to scrutinize how our main modeling assumptions affect the model’s conclusions. In the remainder, we discuss the implications of six major assumptions. While the list is not exhaustive, it is our premise that these assumptions represent the most significant deviations from the reality of medical waiting systems.

Rejection penalty: While our model assumes that patients face no explicit penalty when they refuse a treatment offer, it is common to expect that there are some consequences in practice. There may be a monetary penalty (e.g. penalty for cancellation of an appointment), or an inconvenience cost when such actions have to be justified, sometimes in person. Similarly, patients who refuse treatment may be barred from further opportunities for a fixed amount of time. Extending beyond the realm of Markov models, we also observe the issue of “warnings” in practice: these penalties may become active only after the patient has refused treatment and been warned several times. Although our simple model precludes the analysis of these possibilities, the same methodology can still be applied. The central planner should optimally manage the waiting list by looking ahead and first considering the decision problem of individual patients, which involves formulating either a Markov or a semi-Markov decision model, very much similarly to what we have done. There is a wide taxonomy of options to impose such penalties and we do not intend to adapt our methodology to each one of them, but there is one particular example we wish to consider because it provides insights into the underlying mechanisms at work.

Waiting lists have an inherent bias in favor of patients “in front” over patients “at the back”, and rejection penalties serve to target externality problems by moderating this bias. Intuitively, when privileges are distributed more evenly throughout the waiting list, it is less likely for any individual patient to engage in selfish behavior that acts against the interests of the group. As an extreme case in which all patients are favored equally despite their different positions in line, consider a system in which patients are moved to the end of the line after refusing treatment. As in Section 3, one can write down first-order conditions
for each patient’s optimal decision threshold:

\[
a_n(k) = V_{n-1}(n - 1), \quad k < n
\]

\[
a_n(n) = V_n(n).
\]

These expressions reveal that on the margin, all patients choose between accepting treatment or being moved to the back of the waiting list. For all patients other than the last, refusing treatment implies a continuation payoff of \(V_{n-1}(n - 1)\) because the next patient would accept the marginal treatment, so all these patients face the same set of alternatives.

This example shows that the enforcement of rejection penalties bring about a redistribution of privileges that go beyond the patient’s position in line. While such redistribution may reduce deadweight losses, we can be certain that it serves no purpose in the case of LCFS prioritization, which already attains social optimality.

**Resource degradation:** We next consider the possibility that medical resources may suffer a degradation in quality after being refused. This is especially relevant in the context of kidney transplantation because patient survival rates are highly sensitive to the time between organ procurement and transplantation, which can be expedited by minimizing the number of offer denials. Such degradation is not a cause for concern in the context of our theoretical model because the medical planner has a full understanding of the patients’ decision problem and can make offers directly to the first patient who would accept treatment. Nevertheless, the practical hurdles associated with the implementation of such a system are formidable, and hence social optimality will never be attained, even under LCFS prioritization. Furthermore, the likelihood that a treatment option is refused by all patients is now increased because these options are constantly losing value as they are passed down the waiting list. It follows that patient autonomy causes negative externalities at two levels: existing patients suffer from another patient’s refusal due to resource degradation, and future patients (as in our model) suffer because they face a longer waiting list. Nevertheless, the intuition that prioritization schemes serve as a main lever to control externality problems remains valid, but the claim that LCFS prioritization still minimizes welfare loss may not hold anymore.

**Cost effects:** Our model assumes that capacity is fixed and the medical planner controls the system only through patient prioritization and by holding back some partic-
ular types of service capability. This is a realistic assumption in some instances, such as transplant waiting lists, in which the supply of cadaveric organs is fixed. However, in other circumstances, it may be possible to either expand capacity or improve quality, but at a certain cost. Capacity expansion and improvement decisions govern investments in health care infrastructure, and our analysis suggests that patient autonomy should be held in high regard. The net effect of costs and benefits should be optimized, subject to the constraint that patients are also acting in their own interests. However, this latter constraint is often overlooked in most models of health care investment decisions.

**Imperfect information:** Some readers may challenge our assumption that patients have perfect information about the size of the waiting list. As waiting lists often fluctuate widely on a day-to-day basis, some patients may not even be certain about their position when they are offered an opportunity for treatment, so how can one assume that they have perfect knowledge about the state of the system? While this issue is clearly beyond our modelling framework, we wish to provide some suggestions on how to proceed. There are two possible approaches to tackle this problem.

First, although patients do not know their exact position in the waiting list and queue length, they may maintain a best guess at what they estimate them to be. The learning process by which patients update their guesses based on past observations can be fused together with their decision process to formulate a dynamic program where decisions are based on the imperfect information available. While realistic and conceptually tractable, this approach is highly complicated, especially when patients consider their entire history to draw inferences. Besides, there is no consensus on how their guesses are formed, because possibilities range from Bayesian inference to classical methods like maximum likelihood estimation. Moreover, how can one justify the assumption that patients actually bother to estimate the queue-length and their corresponding position?

The second approach addresses this question by suggesting that patients may simply ignore their position and the queue length in their decision-making process if it is not readily available. This is based upon the assumption that patients are *boundedly rational* - they are content with suboptimal decisions based on information that is readily available when the process of extracting information about the unobservables is too complex and costly;
for a recent review on bounded rationality see Conlisk (1996). In general, the assumption of bounded rationality simplifies the patient’s problem by focussing on a restricted state space, and is thus useful in the study of complex prioritization schemes. However, care has to be taken when adopting this approach because when patients are sufficiently ill-informed, bounded rationality implies that all of them will ignore all information and make the same decisions; this leaves us with a static problem and erodes much of the essence from the analysis. Of course, an important question for future research is whether the problem of patient autonomy is less important when patients are uninformed.

**Homogeneous Patients:** Next, we examine our clearly unrealistic assumption that patients are all of a single type. The conscientious modeler could formulate a separate decision problem for every type of patient under every possible situation, taking into account differences in arrival and reneging rates, inhomogeneous preferences over treatment options, and the detailed state of the system with specifications of the type of patient that is in every position. However, our basic model of the single-type case seems a natural starting point before embarking on this formidable task. Furthermore, there are two well-founded reasons that justify our basic model. First, bounded rationality suggests that patients do not consider massive amounts of information in decision-making; thus, such a high-resolution model may not be necessary at all. Second, most of the relevant information on the characteristics of other patients on the waiting list is not publicly available. Therefore, our basic model suffices to capture the essence of the decision processes of patients on the waiting list. In a companion paper (Su and Zenios, 2002), we have studied a system with multiple classes of patients but without explicit characterization of waiting list dynamics. We leave the examination of the multiclass waiting system as a topic for future research.

We should recognize that not only are there multiple types of patients, but information asymmetries also prevail in medical waiting systems. This implies that our basic model fails to capture another important dimension of strategic behavior: patients have the incentive to misstate their true preferences and health conditions in order to improve individual outcomes. Such behavior exacerbates the strategic problems discussed in this study, and suggests the need to understand and ameliorate the social costs imposed by these information asymmetries. Classical economic theory suggests that incentive-compatible dis-
criminatory control may improve economic efficiency in this case, but such control is often precluded from most health care contexts due to ethical reasons. Even in situations when it is necessary (and possible) to discern between patient types, separate waiting lists may arise for each patient type, in which case our single-type model becomes relevant. This brings us to our next point.

Separate waiting lists: Health care authorities have recently been moving towards maintaining separate waiting lists for patients with different characteristics and preferences. A recent article in _The Guardian_ reports that NHS is working towards maintaining separate waiting lists for different locations and hospitals, and allows patients to choose which list to join. Similarly, in kidney transplantation, Danovitch et al. (2002) note that some organ procurement organizations are establishing a separate waiting list for marginal kidneys, which would expedite the placement of these organs among patients willing to accept them. These measures necessarily imply that patients now face lower variability in treatment options, so that the corresponding incentive problems are reduced. Now, does this shift toward separate queues render futile our preceding analysis of a common queue? No, because our analysis allows us to compare these two alternatives. In OM terminology, separate queues give up the advantages of resource pooling but alleviate the incentive problems that plague a single queue. Gilbert and Weng (1998) has considered this trade-off, but in their model, agency costs result from delegating control to independent managers, rather than the effect of individual versus social optimization that we are considering.

A prototypical multi-server system provides a convenient example to illustrate how to apply our findings to assess this trade-off. Using queueing theoretic terminology, consider a queueing system with two non-identical servers and one customer stream, and where there is a choice between implementing two separate queues or a common queue. For the first option, it is a simple optimization exercise to find the optimal proportion of customers to assign to each queue, and the resulting measure of welfare is readily available. The second option, however, involves a more subtle control problem. Recalling that customers have a preference for the superior server and can choose not to accept service by the inferior server, it is not immediately clear when each server, or both, should operate. Our previous results tell us that it is optimal to keep both servers on, and let customers choose for themselves.
Then, it is not difficult to solve for the threshold position $k^*$ where customers begin to find both servers acceptable: both servers are active when the queue length exceeds $k^*$, but during all other times, the inferior server is idle because customers in line prefer to wait. This completely describes the optimal management of the common queue, which can then be compared to two separate queues. Externality problems could sometimes outweigh the benefits of resource pooling, but the reverse could be true in other cases. Within the framework of our model, our analysis provides a convenient tool to make this difficult choice, which may arise in situations that extend beyond the realm of waiting list management.

10 Concluding Remarks

Health care management is ridden with many subtleties, and the issue of patient autonomy is certainly one of them. On one hand, it is a legal principal and a moral obligation that patients should be free to choose, but on the other hand, this freedom is often exploited when self-serving choices adversely affect social welfare. In this situation, the natural question to ask is: how can we maximize social efficiency while retaining patient autonomy?

In order to study this problem, we have embedded a stochastic game within a queueing model. We find that designing an appropriate prioritization scheme can have a significant impact on social welfare, and explain that it works by reducing the negative externalities of patients’ choices. Although our results show that the externality problems are completely eradicated by using LCFS prioritization, this is usually not acceptable on grounds of social justice and equity. To address this concern, we analyze a continuum of prioritization rules that reflects a trade-off between equity and efficiency. Our intent was for this to serve as a stylized menu of options for policy makers to choose from.

The prototypical example for the medical waiting system studied here is provided by the transplant waiting list. In fact, our results highlight the “law of unintended consequences” that plagues this complex and socially important waiting list. Specifically, in previous work (Zenios, Chertow and Wein, 2000) it has been demonstrated that the current organ allocation system in which a candidate’s priority increases with waiting time is inefficient because organs are not assigned to the candidate most likely to benefit from trans-
plantation. It was also shown that a static priority system is optimal in the sense that it maximizes overall system efficiency. However, the role of patient autonomy has been overlooked in that prior work. On the other hand, our current work demonstrates the role of patient autonomy: even if all transplant candidates are clinically identical, a priority system based on the first-come first-served principle is ineffective because of the negative externalities caused by patient autonomy. By contrast, a system in which patients arriving in the future may receive absolute priority over existing patients can partially alleviate this problem. This suggests that an absolute priority system that gives little emphasis on patient wait will be the most effective for two complementary reasons: not only will it ensure that the most suitable candidates will be assigned these invaluable organs, it will also mitigate the externalities caused by patient autonomy. These point to some fundamental shortcomings of the current organ allocation system that need to be rectified.

In summary, we have developed a medical waiting list model that explicitly takes into account patient preferences and behavior. While our basic model is highly stylized, we have suggested several modifications and extensions that will make it more realistic. We hope that this analysis will stimulate debate on the merits of different priority systems as they apply to the management of medical waiting lists.
Appendix: Proofs

Proof of Proposition 1  The proof shall proceed in two steps.

Step 1: In this step, we use value iteration to establish that the relative value function defined as $\Delta(n) \equiv V(n) - V(n - 1)$ is non-increasing in $n$. Let $V^k(n)$ denote the $k$-th iterate for the value function, with the supremums attained by $\{b^k(n)\}$; the superscript here represents an abuse of notation since in the main part of the paper, superscripts in the value function are used to represent different priority rules. That is, starting arbitrarily with $V^0(n) \equiv 0$, we have, for every $n \geq 0$:

$$V^{k+1}(0) = \frac{\lambda}{\beta + \lambda} V^k(1)$$

$$V^{k+1}(n) = \sup_{b(n) \in \mathcal{B}} \frac{1}{\beta + 1 + \lambda + \gamma n} [hn + F(b(n))(V^k(n - 1) + g(b(n))) + (1 - F(b(n)))V^k(n) + \lambda V^k(n + 1) + \gamma n(V^k(n - 1) + d)]. \quad (19)$$

Next, define for $k \geq 0$ and $n \geq 1$,

$$\Delta^k(n) \equiv V^k(n) - V^k(n - 1),$$

which, by the convergence of the value iteration algorithm, converges to $\Delta(n)$ as $k \to \infty$ for every $n \geq 1$.

Now, in order to establish that $\{\Delta(n)\}$ is non-increasing in $n$, it suffices to show that $\{\Delta^k(n)\}$ is non-increasing in $n$ for every $k \geq 0$. We shall show this by induction. Notice that this holds trivially for $k = 0$. For $k \geq 0$ and $n > 1$, tedious algebra shows that:

$$\Delta^{k+1}(n + 1) = V^{k+1}(n + 1) - V^{k+1}(n)$$

$$\leq \frac{1}{\beta + 1 + \lambda + (n + 1)\gamma} [hn + F(b^k(n + 1))\Delta^k(n)]$$

$$+ (1 - F(b^k(n + 1)))\Delta^k(n + 1) + \lambda \Delta^k(n + 2) + \gamma n \Delta^k(n) + \gamma d] \quad (20)$$

$$\Delta^{k+1}(n) = V^{k+1}(n) - V^{k+1}(n - 1)$$

$$\geq \frac{1}{\beta + 1 + \lambda + (n + 1)\gamma} [hn + F(b^k(n - 1))\Delta^k(n - 1) + (1 - F(b^k(n - 1)))\Delta^k(n)]$$

$$+ \lambda \Delta^k(n + 1) + \gamma (n - 1) \Delta^k(n - 1) + \gamma \Delta^k(n) + \gamma d]. \quad (21)$$
To obtain the inequalities in (20)-(21) we use the fact that $b_k(n+1)$ and $b_k(n-1)$ are suboptimal in state $n$, and we uniformize the transition rates. Inequalities (20) and (21), together with the inductive hypothesis, imply
\[
\Delta^{k+1}(n+1) - \Delta^{k+1}(n) \leq \frac{1}{\beta + 1 + \lambda + (n+1)\gamma} [(1 - F(b_k(n+1)))(\Delta^k(n+1) - \Delta^k(n)) \\
+ F(b_k(n-1))(\Delta^k(n) - \Delta^k(n-1)) + \lambda(\Delta^k(n+2) - \Delta^k(n+1)) \\
+ \gamma(n-1)(\Delta^k(n) - \Delta^k(n-1))] \\
\leq 0. 
\] (22)

Similarly for the boundary terms, we have, for $k \geq 0,$
\[
\Delta^{k+1}(2) \leq \frac{1}{\beta + 1 + \lambda + 2\gamma} [h + F(b_k(2))\Delta^k(1) + (1 - F(b_k(2)))\Delta^k(2) \\
+ \lambda\Delta^k(3) + \gamma\Delta^k(1) + \gamma d] 
\] (23)
\[
\Delta^{k+1}(1) \geq \frac{1}{\beta + 1 + \lambda + 2\gamma} [h + \Delta^k(1) + \lambda\Delta^k(2) + \gamma\Delta^k(1) + \gamma d] 
\] (24)

where (23) follows from taking $n = 1$ in (20), and transition rates are appropriately uniformized. Inequalities (23) and (24), together with the inductive hypothesis, imply
\[
\Delta^{k+1}(2) - \Delta^{k+1}(1) \leq \frac{1}{\beta + 1 + \lambda + 2\gamma} [(1 - F(b_k(2)))(\Delta^k(2) - \Delta^k(1)) \\
+ \lambda(\Delta^k(3) - \Delta^k(2))] \\
\leq 0. 
\] (25)

This completes our inductive proof.

**Step 2:** If we ignore the constraint that $b^*(n) \geq \underline{x}$, then the result follows because the unconstrained optimal decision thresholds are such that
\[
b^*(n) = \Delta(n). 
\] (26)

Tedious algebra can then be used to account for the constraint, but the details are omitted for brevity’s sake.

**Proof of Proposition 2** Consider an arbitrary control policy $\{b(n)\}$, let $\{V_k(n)\}$ and $\{a_k(n)\}$ denote the equilibrium value function and patient decision thresholds. We shall
begin by showing that \{b(n)\} cannot be optimal if there is some \( n \) for which

\[
a_n(n) > V_n(n) \quad \text{and} \quad a_n(n) \neq x. \tag{27}
\]

The proof proceeds by contradiction. Consider the value of \( n \) for which (27) holds. In this case, we must have \( a_n(n) = b(n) \); otherwise, the equilibrium cannot be sustained because a smaller \( a_n(n) \) is feasible and will be chosen instead since \( a_n(n) \) denotes the lowest acceptable quality and \( V_n(n) \) is the continuation payoff from waiting. This similarly implies that if \( b(n) \) is reduced to \( b(n) - \epsilon \), the equilibrium value of \( a_n(n) \) decrease and \( V_n(n) \) will increase. Hence, the policy \{b(n)\} is therefore suboptimal.

This establishes that if \{b(n)\} is optimal, then \( a_n(n) \leq V_n(n) \) or \( a_n(n) = x \) for every \( n \). Since \( a_n(n) < V_n(n) \) is not sustainable in equilibrium, we are left to consider policies with either \( a_n(n) = V_n(n) \) or \( a_n(n) = x \). For any such policy, feasibility implies that \( b(n) = x \) for any \( n \) with \( a_n(n) = x \). For all other values of \( n \), decreasing the control \( b(n) \) to \( x \) does not affect the equilibrium decision thresholds because \( a_n(n) \) already satisfies the first-order condition in (12).

This implies that all these policies yield the same equilibrium outcome as the no-control policy \( b(n) \equiv x \), which is therefore optimal.

\[\blacksquare\]

**Proof of Proposition 3**  This follows as a corollary to Proposition 5, by using \( p = 0 \).  \[\blacksquare\]

**Proof of Proposition 4**  This follows as a corollary to Proposition 5, by using \( p = 1 \).  \[\blacksquare\]

**Proof of Proposition 5**  We shall begin by writing down the optimality equations under randomized absolute priority with parameter \( p \). Let \( a^p(n) \) denote the decision threshold that attains the supremum and \( V^p(n) \) the optimal value function for a patient in position \( n \), we
can express the optimality equations as:

\[
V^p(1) = \frac{1}{\beta + 1 + \lambda + \gamma} [h + \mathcal{F}(a^p(1))g(a^p(1)) + (1 - \mathcal{F}(a^p(1)))V^p(1) \\
+ \lambda(pV^p(2) + (1 - p)V^p(1))] + \gamma d
\]  

\[
V^p(n) = \frac{1}{\beta + 1 + \lambda + \gamma n} [h + \mathcal{F}(a^p(n - 1))V^p(n - 1) + \mathcal{F}(a^p(n))g(a^p(n)) \\
- \mathcal{F}(a^p(n - 1))g(a^p(n - 1)) + (1 - \mathcal{F}(a^p(n)))V^p(n) \\
+ \lambda(pV^p(n + 1) + (1 - p)V^p(n) + \gamma d + (n - 1)\gamma V^p(n - 1))].
\]  

Now, let \( \bar{V}^p(n) = \sum_{i=1}^{n} V^p(i) \). Then, retaining the supremum only for the last term in each summation and substituting \( a^p(n) \) into the other terms, we have:

\[
\bar{V}^p(1) = \sup_{a(1)\in[\underline{a}, \bar{a}(1)]} \frac{1}{\beta + 1 + \lambda + \gamma} [h + \mathcal{F}(a(1))g(a(1)) + (1 - \mathcal{F}(a(1)))\bar{V}^p(1) \\
+ \lambda(p(\bar{V}^p(2) - V^p(1)) + (1 - p)V^p(1))] + \gamma d
\]

\[
\bar{V}^p(n) = \frac{1}{\beta + 1 + \lambda + \gamma} [h + \mathcal{F}(a^p(1))g(a^p(1)) + (1 - \mathcal{F}(a^p(1)))\bar{V}^p(1) \\
+ \lambda(pV^p(2) + (1 - p)V^p(1)) + \gamma d] \\
+ \sum_{i=2}^{n-1} \left\{ \frac{1}{\beta + 1 + \lambda + i\gamma} [h + \mathcal{F}(a^p(i - 1))V^p(i - 1) + \mathcal{F}(a^p(i))g(a^p(i)) \\
- \mathcal{F}(a^p(i - 1))g(a^p(i - 1)) + (1 - \mathcal{F}(a^p(i)))V^p(i) \\
+ \lambda(pV^p(i + 1) + (1 - p)V^p(i) + \gamma d + (i - 1)\gamma V^p(i - 1))] \right\} \\
+ \sup_{a(n)\in[\underline{a}, \bar{a}(n-1)]} \left\{ \frac{1}{\beta + 1 + \lambda + n\gamma} [h + \mathcal{F}(a^p(n - 1))V^p(n - 1) \\
+ \mathcal{F}(a(n))g(a(n)) - \mathcal{F}(a^p(n - 1))g(a^p(n - 1)) \\
+ (1 - \mathcal{F}(a(n)))V^p(n) + \lambda(pV^p(n + 1) \\
+ (1 - p)V^p(n)) + \gamma d + (n - 1)\gamma V^p(n - 1)] \right\}
\]
\[
\begin{align*}
&= \frac{1}{\beta + 1 + \lambda + n\gamma} [h + F(a(1))g(a(1)) + (1 - F(a(1)))V_p(1) \\
&+ \lambda(pV_p(2) + (1 - p)V_p(1)) + \gamma d + (n - 1)\gamma V_p(1)] \\
&+ \sum_{i=2}^{n-1} \left\{ \frac{1}{\beta + 1 + \lambda + n\gamma} [h + F(a(i-1))V_p(i-1) + F(a(i))g(a(i)) \\
&- F(a(i-1))g(a(i-1)) + (1 - F(a(i)))V_p(i) \\
&+ \lambda(pV_p(i+1) + (1 - p)V_p(i)) + \gamma d + (i - 1)\gamma V_p(i-1) + (n - i)\gamma V_p(i)] \right\} \\
&+ \sup_{a(n) \in \mathbb{Z}, a(n-1)]} \left\{ \frac{1}{\beta + 1 + \lambda + \gamma} [hq + F(a(n))g(a(n)) + V_p(n-1) \\
&+ (1 - F(a(n)))V_p(n) + \lambda(p(V_p(n+1) - V_p(1)) + (1 - p)V_p(n)) \\
&+ \gamma nd + \sum_{i=1}^{q} (n - i)\gamma V_p(i) + \sum_{i=1}^{q} (i - 1)\gamma V_p(i-1)] \right\} \\
&= \sup_{a(n) \in \mathbb{Z}, a(n-1)]} \left\{ \frac{1}{\beta + 1 + \lambda + \gamma} [hn + F(a(n))g(a(n)) + F(a(n))\bar{V}_p(n-1) \\
&+ (1 - F(a(n)))V_p(n) + \lambda(p(V_p(n+1) - V_p(1)) + (1 - p)V_p(n)) \\
&+ \gamma nd + \gamma n\bar{V}_p(n-1)] \right\} \\
&\text{(32)}
\end{align*}
\]

Next, let \( J(n) = z + \bar{V}_p(n) \) where \( z = \frac{\lambda p}{\beta} V_p(1) \). Then, from the expression for \( \bar{V}_p(1) \) in (30), and after some tedious algebra we have:

\[
\begin{align*}
J(1) &= \sup_{a(1) \in \mathbb{Z}} \left\{ \frac{1}{\beta + 1 + \lambda + \gamma} [h + F(a(1))(g(a(1)) + \lambda p \frac{p}{\beta + \lambda p} J(1) \\
&+ (1 - F(a(1)))J(1) + \lambda(pJ(2) + (1 - p)J(1)) + \gamma d + \frac{\lambda p}{\beta + \lambda p} J(1)] \right\} \\
&\text{(34)}
\end{align*}
\]

Similarly, using the expression for \( \bar{V}_p(n) \) in (33), we have:

\[
\begin{align*}
J(n) &= \sup_{a(n) \in \mathbb{Z}, a(n-1)]} \left\{ \frac{1}{\beta + 1 + \lambda + \gamma} [hn + F(a(n))g(a(n)) + F(a(n))J(n-1) \\
&+ (1 - F(a(n)))J(n) + \lambda[pJ(n+1) + (1 - p)J(n)] + \gamma nd + \gamma nJ(n-1)] \right\} \\
&\text{(35)}
\end{align*}
\]
After removing “dummy” transitions which were introduced purely for purposes of uniformization, (34)-(35) become

\[ J(1) = \sup_{a(1) \in [x, x]} \frac{1}{\beta + 1 + \lambda p + \gamma} \left[ h + \overline{F}(a(1))(g(a(1)) + \frac{\lambda p}{\beta + \lambda p} J(1)) \right] \\
+ \left[ (1 - \overline{F}(a(1))) J(1) + \lambda p J(2) + \gamma (d + \frac{\lambda p}{\beta + \lambda p} J(1)) \right] \]

\[ J(n) = \sup_{a(n) \in [x, a(n-1)]} \frac{1}{\beta + 1 + \lambda p + \gamma n} \left[ h n + \overline{F}(a(n)) g(a(n)) + \overline{F}(a(n)) J(n-1) \right] \\
+ \left[ (1 - \overline{F}(a(n))) J(n) + \lambda p J(n+1) + \gamma nd + \gamma n J(n-1) \right] \]  

(36)

The monotonicity result of Proposition 1 implies that imposing a constraint that \(a(n)\) is decreasing in \(n\) does not change the solution of optimality equations (36)-(37). Therefore,

\[ J(0) = \frac{\lambda}{\beta + \lambda} J(1) \]

\[ J(1) = \sup_{a(1) \in [x, a]} \frac{1}{\beta + 1 + \lambda + \gamma} \left[ h + \overline{F}(a(1))(J(0) + g(a(1))) \right] \\
+ \left[ (1 - \overline{F}(a(1))) J(1) + \lambda J(2) + \gamma (J(0) + d) \right] \]

\[ J(n) = \sup_{a(n) \in [x, a(n-1)]} \frac{1}{\beta + 1 + \lambda + \gamma n} \left[ h n + \overline{F}(a(n))(J(n-1) + g(a(n))) \right] \\
+ \left[ (1 - \overline{F}(a(n))) J(n) + \lambda J(n+1) + \gamma n (J(n-1) + d) \right] \]  

(39)

Therefore, the optimal decision thresholds \(\{a^p(n)\}\) must solve (1)-(2) with the arrival rate modified at \(\lambda p\).

Proof of Proposition 6  The proof shall proceed in three steps. In the first step, we consider the relative value function for the optimality equations (1)-(2), defined by \(\Delta(n) \equiv V(n) - V(n-1)\) and \(\Delta(0) \equiv V(0)\), and show that it solves a linear program. In the second step, we consider the relative value function when the arrival rate is \(\lambda'\) and show that it is feasible for the linear program corresponding to \(\lambda < \lambda'\). This, in turn, implies that the relative value function is decreasing in \(\lambda\). The claim is then established in step 3 based on the argument presented in the proof of Proposition 1.

Step 1: Let us express the optimality equations (1)-(2), in terms of the relative value function.
functions.

$$\beta \Delta(0) = \lambda \Delta(1)$$  \hspace{1cm} (41)

$$\beta V(n) = \sup_{b(n) \in [\underline{x}, \overline{x}]} [hn + \overline{F}(b(n))(g(b(n)) - \Delta(n)) + \lambda \Delta(n + 1) + \gamma n (d - \Delta(n))], \forall n \geq 1$$  \hspace{1cm} (42)

Using (41) and the fact that $V(n) = \sum_{i=0}^{n} \Delta(i)$, we can write this as: $\forall n \geq 1$,

$$0 = \sup_{b(n) \in [\underline{x}, \overline{x}]} [hq + \overline{F}(b(n))(g(b(n)) - \Delta(n)) + \gamma n (d - \Delta(n)) + \lambda (\Delta(n + 1) - \Delta(1)) - \beta \sum_{i=1}^{n} \Delta(i)]$$  \hspace{1cm} (43)

It then follows that the optimal relative value functions must solve the following linear program (see Puterman(1994)), where $\{w(n)\}$ are arbitrary positive constants:

$$\min_{\{\Delta(n)\}} \sum_{n=1}^{\infty} w(n) \Delta(n)$$  \hspace{1cm} (44)

st $0 \geq [hn + \overline{F}(b(n))(g(b(n)) - \Delta(n)) + \gamma n (d - \Delta(n))$

$$+ \lambda (\Delta(n + 1) - \Delta(1)) - \beta \sum_{i=1}^{n} \Delta(i)] \quad \forall b(n) \in [\underline{x}, \overline{x}], \forall n \geq 1.$$

We shall call this linear program $(LP; \lambda)$.

**Step 2:** Consider now the relative value function $\Delta'(n)$ that corresponds to an arrival rate $\lambda' > \lambda$. We shall show that $\Delta'(n) \leq \Delta(n)$ for every $n$.

The optimal relative value functions $\{\Delta(n)\}$ and $\{\Delta'(n)\}$ must solve $(LP; \lambda)$ and $(LP; \lambda')$ respectively. Since we have previously shown that $\Delta(n)$ is non-increasing in $n$, it is not difficult to see that $\{\Delta(n)\}$ is feasible in $(LP; \lambda')$. This is because the right-hand-side of (45) remains negative when $\lambda$ is replaced by $\lambda' \geq \lambda$.

Now, suppose that there is some $n$ for which $\Delta'(n) > \Delta(n)$. Then, we can choose positive weights $\{w(n)\}$ such that $\sum_{i=1}^{n} w(i) \Delta'(i) > \sum_{i=1}^{n} w(i) \Delta(i)$, which contradicts the optimality of $\{\Delta'(n)\}$ in $(LP; \lambda')$. Therefore, we must have $\Delta'(n) \leq \Delta(n)$ for every $n$.

**Step 3:** Finally, we can show that $b'(n) \leq b(n)$ for every $n$. This follows directly from Step 2 using the same argument provided in the proof of Proposition 1.
References


