Initial Draft

UNIQUE ESSENTIAL COMPONENT OF EQUILIBRIA OF A SINGLE-ITEM FIRST-PRICE AUCTION WITH PRIVATE VALUES

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1. Introduction

The literature on auctions has focused on establishing existence of equilibria and characterizing particular equilibria. Krishna [2] provides a comprehensive exposition. The issue of uniqueness of equilibrium has not been resolved except for some cases of symmetric equilibria of auctions with symmetric players. The prominent exception is the article by Lebrun [3], who establishes uniqueness of the equilibrium of a first-price auction of a single item in which players have independent private values, provided the distribution functions of players' values are strictly log-concave at the highest of the lower bounds of their supports.

In this paper we extend Lebrun's result to the case of private values whose joint distribution has a positive density on a hypercube. We use the Leray-Schauder index of a component of equilibria to establish that there is a unique essential component, i.e. one such that every nearby fixed-point map has a nearby fixed point. An intermediate step of the proof uses the index for an equilibrium component of a finite game to establish that also a finite game that approximates the auction game has a unique essential component; moreover, in this case it is known [1] that an essential component is uniformly hyperstable and every nearby game has a nearby equilibrium. One can interpret our result as showing that the graph of essential equilibria has the form illustrated schematically in Figure 1. Not shown in the figure are possible additional equilibria that are inessential and thus have index zero.

2. A First-Price Auction

Consider a first-price auction for a single item. The set of bidders is N. Bidders can be asymmetric. Acting simultaneously, each bidder n observes his value v_n and then chooses a bid $\beta_n(v_n)$. Then the item is assigned randomly to one of the bidders choosing the highest bid.

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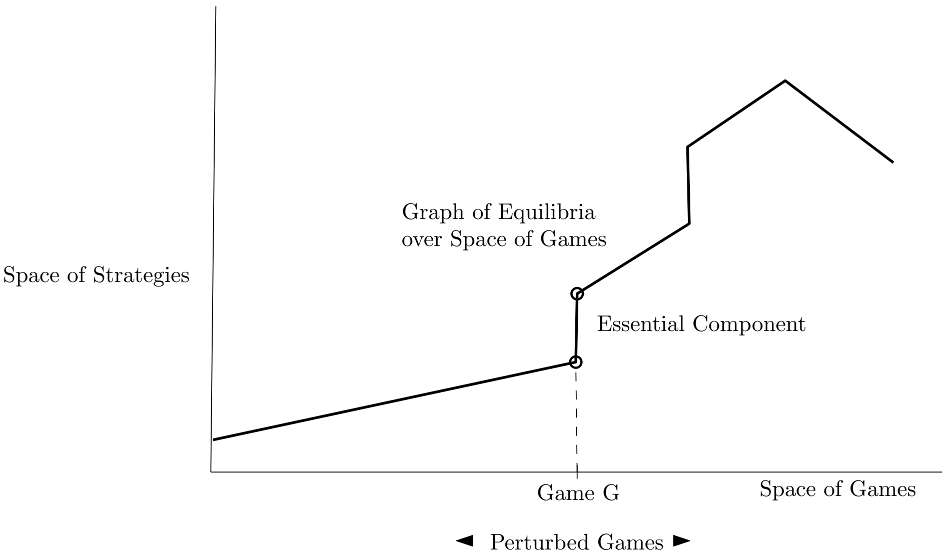

FIGURE 1. Schematic illustration of the equilibrium graph.

We assume it is common knowledge that the vector $v = (v_n)_{n\in N}$ of bidders' values of the item is drawn from the joint distribution $\mathcal{F}$ that has a positive density on V^N, where $V \subset \mathbb{R}$ is an interval. Moreover, for each bidder $n \in N$:

- his utility function $u_n : \mathbb{R} \to \mathbb{R}$ is continuous and strictly increasing, and $u_n(0) = 0$,
- his payoff when his value is v_n and he bids $\beta_n(v_n)$ is $u_n(v_n - \beta_n(v_n))$ if he wins the item and 0 otherwise.

For each bidder n a pure strategy is a function $\beta_n : V \to V$, where $\beta_n(v_n)$ is his bid when his value is v_n. If $\beta = (\beta_n)_{n\in N}$ is a profile of bidders' pure strategies then n's expected payoff is

$$U_n(\beta) = E\left[u_n(v_n - \beta_n(v_n)) \prod_{m\neq n} \mathbf{1}_{\beta_n(v_n)\geqslant\beta_m(v_m)}\right].$$

Note the implicit assumption that ties have zero probability.[1]

This specification defines a game $\mathcal{G}$ in normal form.

3. AN APPROXIMATE GAME

Define a finite game $\hat{G}$ that approximates $\mathcal{G}$ as follows. For each bidder n let V_n be a finite subset of V, and for each $v \in V_n$ let A_{nv} be a finite subset of V. Assume that $A_{nv} \cap A_{n'v'} = \emptyset$ if $n \neq n'$. In $\hat{G}$, values are drawn from the piecewise-linear approximation F of $\mathcal{F}$ that agrees with $\mathcal{F}$ at values in $\prod_n V_n$. Moreover, for each bidder n:

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[1] Alternatively, all high bidders receive an item but this event has zero probability. See Remark 5.2 below.

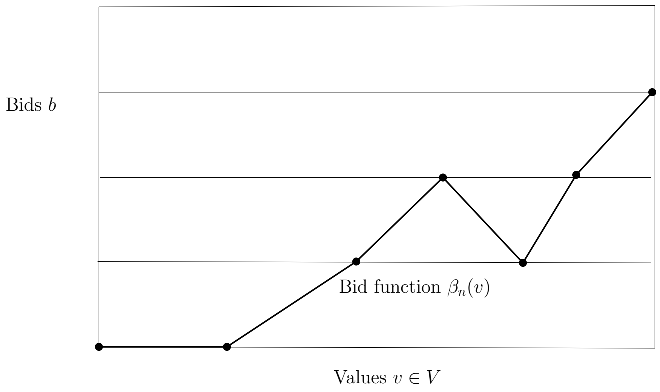

FIGURE 2. Schematic illustration of an interpolated bid function.

- Bidder n can choose his bid β_{nv} only for values $v \in V_n$.
- For each value $v \in V_n$ he must choose a bid in A_{nv}.
- His bid function is piecewise linear; viz., for each value $v \in V \setminus V_n$ his bid β_{nv} is obtained by linear interpolation between his bids in A_{nv} at the two adjacent values in V_n (and constant outside this domain)[@?]. See Figure 2.

In all other respects the rules and payoffs of $\hat{G}$ are the same as in Section 2. This specification defines a finite game $\hat{G}$ in normal form.

Here we consider the game G that is the agent-normal-form of $\hat{G}$. In G each pair (n, v) with $v \in V_n$ is treated as a separate player denoted nv, and all other agents are dummy players since they have no actions.

4. UNIQUE ESSENTIAL EQUILIBRIUM COMPONENT OF THE APPROXIMATE GAME

Theorem 4.1. *The game G has a unique essential component of equilibria.*

Proof. Because the space of mixed strategies is contractible, the sum of the indexes of all components is necessarily $+1$. Recall that a component of equilibria is essential iff its index is nonzero [1]. Therefore it suffices to show that every component of equilibria has nonnegative index.

A perturbation $G \oplus g$ of G gives a bonus $g_{nv,b}$ to player nv for using his strategy $b \in A_{nv}$. There exists a generic perturbation $G \oplus g^0$ of G, where each $\|g^0_{nv}\|$ is arbitrarily small, such that all equilibrium components are singletons.

Let σ^0 be any equilibrium of $G \oplus g^0$. We construct a homotopy parameterized by $t \in [0, 1]$ that continuously deforms $G \oplus g^0$ through similar games $G^t \oplus g^t$, and that deforms σ^0 through equilibria σ^t of $G^t \oplus g^t$, i.e. the homotopy moves $(G^t \oplus g^t, \sigma^t)$ within the graph of equilibria.

Define the homotopy as follows. For each player nv, the homotopy deforms G to G^t by deforming each A_{nv} to $A^t_{nv} = \{[1-t]b + t\bar{b}^0_{nv} \mid b \in A_{nv}\}$, where $\bar{b}^0_{nv}$ is the expectation of nv's bid according to his mixed strategy σ^0_{nv} in the equilibrium σ^0 of $G \oplus g^0$. The homotopy deforms σ^0 to σ^t by letting $\sigma^t_{nv}([1-t]b + t\bar{b}^0_{nv}) = \sigma^0_{nv}(b)$ for each $b \in A_{nv}$. Finally, the homotopy deforms g^0 to g^t such that σ^t is an equilibrium of $G^t \oplus g^t$. This latter deformation adjusts the payoff from each pure strategy in the support of σ^t_{nv} so that it remains optimal, and reduces payoffs from others so that they do not become optimal.

To verify that this is a valid homotopy, observe that g^t can indeed be chosen continuously so that σ^t is an equilibrium of $G^t \oplus g^t$. This follows from the property of auctions with private values that player nv's optimal bids depend on others' strategies only via the distribution functions of their bids. The deformations A^t and σ^t continuously alter the distribution of bids by each player $n'v'$ with $n' \neq n$, which continuously alters the probability of winning and thus the expected payoffs to player nv. This alteration in the probability of winning can be compensated by continuous variation of the bonuses $(g^t_{nv,b})_{b \in A_{nv}}$ so that if $g^t_{nv,b}$ is added as a bonus to using strategy $[1-t]b + t\bar{b}^0_{nv} \in A^t_{nv}$ then σ^t is an equilibrium of $G^t \oplus g^t$.

Observe that at $t = 1$ the equilibrium σ^1 is an pure-strategy equilibrium, since each agent nv puts all probability on the bid $\bar{b}^0_{nv}$. Moreover, it is isolated since no other bid is optimal.

Because the index of any isolated pure-strategy equilibrium such as σ^1 is nonnegative, and the index is invariant to homotopic deformations, this implies that the index of σ^0 is also nonnegative. Because the sum of the indexes of those equilibria of $G \oplus g^0$ near each equilibrium component C of G is the index of C, this implies that the index of each component C is nonnegative. Since the sum of the indexes of all equilibrium components is +1, and each essential component has nonzero index, there can be only one essential component and its index is +1. ☐

5. Unique Essential Equilibrium Component of the Auction

We now derive the implications for the original auction game $\mathcal{G}$.

Theorem 5.1. *The auction game $\mathcal{G}$ has a unique essential equilibrium.*

Proof. We use the property of the Leray-Schauder index for contractible spaces of bid functions that:

- The sum of the indices of all components of equilibria of $\mathcal{G}$ is +1.
- For every finite-dimensional game G that approximates $\mathcal{G}$ sufficiently closely, the sum of the indexes of those equilibrium components of G near a component $\mathcal{C}$ of $\mathcal{G}$ is the index of $\mathcal{C}$.

- A component is essential iff its index is nonzero.

The normal and agent-normal forms of $\mathcal{G}$ are equivalent. For each essential component $\mathcal{C}$ of equilibria of (the agent-normal form of) $\mathcal{G}$, each sufficiently close approximate game G as defined in Section 3 is finite dimensonal and, according to Theorem 4.1, has a unique essential equilibrium component, and its index is $+1$. Therefore, every essential component of the equilibria of $\mathcal{G}$ has positive index. Since the indexes of the essential components are positive, and their sum is $+1$, $\mathcal{G}$ has only one essential component and its index is $+1$. $\square$

Remark 5.2. The formulation in Section 2 treated ties has having zero probability. An alternative formulation supposes that every high bidder wins an item. In this case there is an equilibrium of $\mathcal{G}$ in which, for example, all players invariably bid b. But such equilibria are inessential; viz. if the game is perturbed by giving a small bonus $\varepsilon > 0$ for submitting the bid $b + \varepsilon$ then this equilibrium disappears because it becomes preferable to bid $b + \varepsilon/2$ when all others bid b.

References

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