Design of Efficient Trading Procedures

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This paper describes a formulation of the problem of mechanism design when the participants have private information. Allocations that are efficient within the constraints imposed by incentive compatibility and individual rationality are characterized in terms of necessary conditions. The formulation includes as special cases the design of optimal trading procedures, such as bargaining and auctions.

One motive for economic analysis is to improve efficiency. Organizational designs, contract negotiations, and market trading procedures are significant arenas for improvements. In recent years, analyses of these topics have used new theories of mechanism design and implementation. In general, this work aims to construct for each economic environment the rules of a game that yields an efficient outcome when the participants use equilibrium strategies. In this sense the game is efficient, compared to other games that could be used; or the contract or organizational structure that embodies the rules is efficient.

The design of procedural rules invokes a different and weaker criterion for efficiency than the traditional criterion of Pareto optimality. The Pareto criterion requires the strong property that no alternative outcome could improve the welfare of every participant. How the outcome is achieved is moot; for example, a Walrasian allocation is supposedly an efficient outcome, but the issue of how markets are organized to determine prices and trades is not addressed. In contrast, the design approach includes practical matters of implementation. This implies, for instance, that monopoly profits may be unavoidable in a market with few traders, and therefore the objective of the design is to minimize the distortionary effects of the participants’ influence on prices. This modification of the

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The Pareto criterion is sometimes called second-best optimality.

A further aspect of the design approach is that it takes explicit account of each participant’s private information, including his preferences as well as other economically relevant data. That is, designs are constrained by practical limitations: some information cannot be observed by others, and some actions cannot be monitored or verified. Some environments admit mechanisms that encourage participants to reveal their private information, but usually the incentives required to promote truthful revelation impose costs in the form of Pareto-inefficient outcomes. That is, like monopoly rents, informational rents are often inescapable. The design objective is again to minimize the distortionary effects of these rents. The requirement that sufficient incentives must be provided to induce participants to reveal private information, or to undertake specific actions, is called the incentive compatibility constraint on the design.

A standard application is contracting between a principal and an agent, in which the agent’s information and action are unobservable: in this case the principal’s reward to the agent can be based only on observables such as output, and the agent’s action (depending on his information) presumably serves his own interests as modified by the schedule of output-contingent rewards offered by the principal. The same features recur in markets. In an auction, for instance, bidders with private information about the value of the item for sale can obtain informational rents: in equilibrium each bids less than his estimate of the value, although this effect is diminished if there are many bidders.

The formulation of efficiency criteria that recognize incentive compatibility constraints is due to Holmström and Myerson (1983). They define three forms that are increasingly restrictive. To state their criteria it is useful to identify each participant’s private information with his type, and to apply each criterion to outcomes that are contingent on the list of all their types. Each such list is one of the possible states of the environment, and an allocation is a function that assigns to each list an outcome contingent on that list, subject to the proviso that the allocation is implementable — in the sense that it is achievable by some procedure consistent with incentive compatibility. The criteria differ according to whether they are invoked contingent on the list itself, contingent on the possible types of each participant, or a priori before any participant obtains private information. The criterion for ex post efficiency says that an allocation is efficient if no other implementable allocation improves each participant’s list-contingent welfare for every list that might occur. The criterion for interim efficiency says that an allocation is efficient if no other allocation improves the type-contingent expected welfare of every
participant for every type he might be. Lastly, the criterion for *ex ante* efficiency requires that no other allocation improves the expected welfare of every participant, calculated by taking the expectation over the types he might become. In practice, these criteria can be implemented by using welfare weights for the participants that are list-contingent, type-contingent, or constant. An efficient design is therefore one that maximizes the expectation of the weighted sum of the participants’ expected benefits, subject to the incentive compatibility constraints and any other feasibility constraints imposed by the environment. Thus, the design problem is a special kind of constrained optimization problem, as we illustrate in §1.

*Ex post* efficiency is rarely invoked because it is a very weak criterion; for example, it ignores the benefit that one participant could gain from insuring against what others’ types might be. *Interim* efficiency applies to environments in which participants’ types are fixed data, though each participant’s type is unknown to others. For example, this is usually the case after they have arrived at a market or contract negotiation. The strongest criterion, *ex ante* efficiency, is useful to design organizational structures or market procedures in anticipation that they will be used later in many different environments that will differ according to the types of the participants involved.

In many cases there is a specific feasibility constraint that is usefully treated in a special way. If a participant has an outside option enabling him to refuse participation in the procedure, then the procedure must assure him benefits no less than what he could get from the outside option. This is the usual case in organizational design because there is an exogenous labor market, and in exchange markets because one can refuse to trade. Participation constraints are also called individual rationality constraints. Their restrictiveness is inverse to the ordering of efficiency criteria. A participation constraint is *ex ante*, *interim*, or *ex post* according to whether the outside option can be exercised only before knowing one’s type, after knowing one’s type, or at the conclusion of the procedure. An *ex post* participation constraint is highly restrictive, since it enables one to reject his list-contingent outcome if it is inferior to his outside option; nevertheless, this is the case in some markets because each trader can refuse exchange even after other traders’ information has been revealed by their bids. Many auctions, however, require bidders to commit to purchasing at prices no more than their offered bids (e.g., they submit sealed-tenders that represent contractual commitments), and in this case an *interim* constraint is more descriptive. In §1 we use an *interim* participation constraint, but the methods are adaptable to other formulations.
Much of mechanism design theory relies on a simple proposition known as the revelation principle. In practice the design prescribes an implementation in the form of a game whose rules specify the actions available to each participant and the outcome that results from each combination of actions they might choose. This is then implemented as an equilibrium in which each participant chooses an optimal strategy, namely a specification of his action depending on his information. Thus, the outcome is obtained via the translation from types to actions, and then from actions to the outcome. From an abstract perspective, however, this is merely a single mapping from types to the outcome, namely an allocation as defined above. Thus, it suffices to adopt a formulation in which the designer chooses a rule for assigning an allocation to each list of types. This leaves unsolved the task of designing a practical implementation, but it provides a convenient test of the efficiency of each implementation that might be proposed; in addition, the form of an efficient allocation rule often provides clues about how to construct a game that implements it.

To ensure that an allocation rule is incentive compatible one uses the requirement that each participant must have an incentive to reveal truthfully his private information. This can be appreciated intuitively by supposing that an implementation is actually used: in this case a participant essentially tells the manager of the procedure to use his reported information and his equilibrium strategy to compute the optimal action used in the game. Because the type-contingent action is optimal, a false report cannot obtain better expected benefits, and therefore truthful reporting is optimal. A game in which each participant’s equilibrium strategy is merely to report truthfully his private information is called a direct revelation game.¹

These ingredients can be summarized as follows for the case that the efficiency criterion and the participation constraints are formulated in their interim forms. The objective is to maximize a weighted sum of the participants’ expected utilities by choosing an allocation, represented as a function specifying the outcome for each participant as a function of the entire list of their reported types. Each participant’s weight can be contingent on his reported type. The constraints that restrict the choice of allocation require incentive compatibility in the form of incentives for truthful reporting of one’s type, and individual rationality in the form of assurance that each participant’s type-contingent outside option

¹ Stronger results can be obtained in some cases by exploiting the dynamic structure of the game and insisting on stronger equilibrium criteria, such as subgame perfection or exclusion of weakly dominated strategies, but we do not address these amendments here.
is not preferred to his type-contingent outcome from the allocation. The solution of this maximization problem defines the outcome function for a direct revelation game that implements an interim efficient allocation. As mentioned, this formulation bypasses the practical problem of implementing the allocation as a game other than direct revelation; e.g., in an exchange context the direct revelation game represents the reduced form of the rules of an auction and the equilibrium bid and offer strategies used by the buyers and sellers.

This formulation is representative of the constructions developed in the work of Myerson (1979, et seq.) and subsequent authors who emphasize the design of efficient trading procedures. In addition, it encompasses a large literature on regulation, organizational design, contracting, and negotiation that focuses on principal-agent relationships affected by adverse selection and moral hazard.

Most of this literature imposes restrictive assumptions to get clean characterizations of sufficiency conditions. The basic methodology is widely applicable, however, to obtain necessary conditions for general formulations. The key to this development is an article by Mirrlees (1976a) that shows how the Divergence Theorem from multivariate calculus simplifies the derivation. In this chapter we employ Mirrlees’ method to present a capsule summary of necessary conditions that characterize an efficient allocation. As the setting for a general formulation, we use the context of selection of an investment project by a group of investors, each of whom might be risk averse, have private information, etc. In this setting the problem emphasizes efficient selection of the project and subsequent sharing of the profits. By a simple reinterpretation, however, this formulation applies equally to a principal-agent relationship in which the project is the action undertaken by the agent in response to the incentive represented by the share of profit. The formulation also applies to the design of efficient trading procedures, such as auctions, by interpreting the project as the set of buyers and sellers who trade and the resulting prices and quantities, as we illustrate in §4.

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3 See Baron (1989), Hart and Holmström (1987), and Kennan and Wilson (1992) for surveys of these literatures. Palfrey (1992) surveys the more general literature on implementation via Bayesian games of incomplete information.

investment, contracting, and market trading.

The formulation allows two features often excluded in standard formulations: multidimensional correlated private information and nonlinear utilities representing risk aversion. The formulation is static so we cannot address implementations that exploit equilibrium refinements of extensive-form games.\(^5\) No account is taken of moral hazard associated with private actions until a partial formulation is provided in \(\S 3\).

1. Formulation

Consider a group of individuals indexed by \(i = 1, \ldots, m\) that is choosing one among several risky projects indexed by \(j = 0, 1, \ldots, n\). The choice also requires a plan for allocating the proceeds among the members. If the \(j\)-th project is chosen then the group’s net income will be \(y_j(\theta)\). This income is a random variable defined on a probability space \((T, T, F)\) with generic element \(\theta \equiv (t, \tau)\), where \(t \equiv (t_1, \ldots, t_m)\). As described below, the variable \(t_i\) represents the \(i\)-th individual’s private information, while \(\tau\) comprises all other stochastic features that are not observed by any member of the group until after all decisions have been taken; we omit specification of other features that are common knowledge. The group’s decision rule is a pair \((x, s)\) specifying the probabilities \(x \equiv (x_j)\) assigned to the various projects and the sharing rule \(s \equiv (s_{ij})\) that allocates income among the individuals depending on the project and its outcome.

We suppose that the \(i\)-th individual has private information about the realization of the \(i\)-th component \(t_i\) of \(t\); we say that \(t_i\) is his type. Consequently, the decision rule can depend on reports \(\hat{\ell} \equiv (\hat{\ell}_1, \ldots, \hat{\ell}_m)\) about their types submitted by the members. In addition, the sharing rule can depend on the subsequently observed realization of a random variable \(z_j(\theta)\) depending on the project chosen; naturally the income \(y_j\) is measurable with respect to the outcome \(z_j\), say via \(g_j(z_j)\). Thus, the probability \(x_j(\hat{\ell})\) assigned to the \(j\)-th choice and \(i\)’s share \(s_{ij}(z_j; \hat{\ell})\) depend on the list \(\hat{\ell}\) of reports, and the shares depend further on the observed outcome. These functions must satisfy the feasibility conditions that

\[
x_j(\hat{\ell}) \geq 0 \quad \text{and} \quad \sum_j x_j(\hat{\ell}) = 1, \quad \text{and} \quad \sum_i s_{ij}(z_j; \hat{\ell}) \leq g_j(z_j)
\]

(assuming free disposal of income), for all reports \(\hat{\ell}\), choices \(j\), and likely outcomes \(z_j\).\(^6\)


\(^6\) Some formulations allow the weaker condition \(\mathcal{E} \left\{ \sum_i s_{ij} - g_j \right\} \leq 0\) on the pre-
In practice, it may be necessary to impose lower bounds on the individuals’ shares, say $s_{ij} \geq s^*_i$; we mention later how these are included.

Individual $i$ has preferences specified by a state-contingent utility function $u_{ij}(\cdot; \theta)$ that depends on the project and his share of the income. The functions $\{u_{ij}(\cdot; \cdot)\}$ as well as the probability distribution $F$, including its various conditional distributions, are assumed to be common knowledge among the members. Consequently, given any profile $\sigma$ of reporting strategies $\sigma_i : t_i \mapsto \hat{t}_i$, member $i$’s conditional expected utility is

$$\hat{U}_i(t_i; \sigma) = \mathbb{E} \left\{ \sum_j x_j(\sigma(t))u_{ij}(s_{ij}(z_j(\theta); \sigma(t)); \theta) \mid t_i \right\}.$$

Due to the revelation principle, there is no loss of generality in assuming that the decision rule is designed to induce accurate reporting by the members; that is, $\sigma$ is the identity function $I$. For this case, define $U_i(t_i) \equiv \hat{U}_i(t_i; I)$.

To ensure accurate reporting the decision rule must include sufficient incentives. A decision rule is 	extit{incentive compatible} if accurate reporting is a Nash equilibrium; that is, if any one member expects others to report accurately then accurate reporting is one of his optimal responses. We also include 	extit{individual rationality} or participation constraints, requiring that

$$U_i(t_i) \geq U^*_i(t_i)$$

for each $i$ and $t_i$, where $U^*_i(t_i)$ represents a minimal expected utility that member $i$’s type $t_i$ must obtain. The 	extit{feasible} decision rules are those that are incentive compatible and individually rational.

By convention, $j = 0$ usually signifies a riskless null project (disband the group) that provides this utility directly: $u_{i0}(s; \theta) = U^*_i(t_i)$ and $y_0(\theta) = 0$. This project is one that any member can insist on, which makes meaningful the notion that other projects

\footnote{An exposition of the revelation principle is in Myerson and Satterthwaite (1983), among others.}

\footnote{The formulation can be adapted to participation constraints that impose lower bounds either \textit{ex post}, say $\sum_j x_j u_{ij} \geq U^*_i(t)$ for each $t$, or \textit{ex ante}, say $\mathbb{E}\{U_i(t_i)\} \geq U^*_i$.}

\footnote{For expositions of the concepts of incentive compatibility, individual rationality, and \textit{interim} and \textit{ex ante} incentive efficiency, see Holmström and Myerson (1983). They define a stronger concept of durability that we do not address here. These constructions differ from d’Aspremont and Gérard-Varet (1979, 1982) mainly in the imposition of individual rationality constraints.}
require unanimous consent of the members. In some formulations it is more accurate descriptively to suppose that each individual \( i \) has a personal action (e.g., refusal to trade) that he can take unilaterally to ensure that his utility is \( U_i^*(t_i) \). This difference is immaterial since in any case all that matters is that each individual \( i \) must be assured utility at least \( U_i^*(t_i) \). The participation constraint can be strengthened to coalitional rationality, requiring that the decision rule is in the core of the associated cooperative game as in Wilson (1978), but we omit this complication here.

A feasible decision rule is incentive efficient if it is common knowledge among the members that no other feasible decision rule would benefit some without harming others. In technical terms, a decision rule is incentive efficient if there exist nonnegative type-contingent welfare weights \( \{ \lambda_i(t_i) \} \), not all zero for any state \( t \), such that the rule maximizes the welfare measure \( W = E \{ \sum_i \lambda_i(t_i)U_i(t_i) \} \) on the domain of feasible rules. If these weights are not type-contingent, then the rule is ex ante incentive efficient. Individual rationality and stronger participation constraints such as coalitional rationality can be interpreted as imposing constraints on the allowable welfare weights.

Our objective in the subsequent analysis is to characterize the incentive-efficient decision rules for a general class of models. This requires several regularity assumptions.

1. A feasible decision rule exists (e.g., selection of the null project).
2. The net income \( y_j(\theta) \) or \( g_j(z_j) \) from each project is a real vector of dimension \( \ell \).
3. Each individual \( i \)'s type \( t_i \) is a real vector of dimension \( K_i \). The conditional support of \( t \) is a rectangle \( D = \{ t \mid a \leq t \leq b \} \) that is independent of \( \tau \). Let \( D = D_1 \times \cdots \times D_m \), where \( D_i = \{ t_i \mid a_i \leq t_i \leq b_i \} \subset \mathbb{R}^{K_i} \) is the support of \( i \)'s type, and use \( \partial D_i \) to denote the boundary of \( D_i \).
4. The joint conditional distribution \( F(t \mid \tau) \) and the marginal distributions \( F_i(t_i) \) have density functions \( f(t \mid \tau) \) and \( f_i(t_i) \) that are positive and continuously differentiable with uniformly bounded derivatives on the domains \( D \) and \( D_i \). For later reference, define
   \[
   \phi_i(\theta) \equiv \frac{\partial f(t \mid \tau)/\partial t_i}{f(t \mid \tau)} - \frac{\partial f_i(t_i)/\partial t_i}{f_i(t_i)},
   \]
   where \( \theta \equiv (t, \tau) \) and the indicated partial derivatives are gradient vectors. Note that \( E \{ \phi_i(\theta) \mid t_i \} \equiv 0 \).
5. Each utility function \( u_{ij} \) is bounded, increasing, and concave with respect to the share \( s \) and continuously differentiable with uniformly bounded derivatives with respect to the share and \( i \)'s type \( t_i \). For later reference, define the gradient vector
   \[
   v_{ij}(s; \theta) \equiv \partial u_{ij}(s; t, \tau)/\partial t_i.
   \]
These assumptions suffice, via the Lebesgue dominated convergence theorem, to assure that differentiation with respect types can be permuted with respect to expectation over other random variables, which we use repeatedly. Assumptions 3 and 4 exclude the possibility that the support of one member’s type is restricted by the observation of others’ types. In particular, they exclude the case that some members have inferior information (e.g., \( t_1 \) is \( t_2 \)-measurable) or no private information, but the formulation can be amended to address this special case.\(^{10}\)

We use the notation included in assumptions 4 and 5 to specify the requirement of incentive compatibility in a convenient form. Incentive compatibility requires for each member \( i \) and each type \( t_i \) that

\[
U_i(t_i) = \max_{\sigma} \tilde{U}_i(t_i; \sigma)
\]

where the possible reports \( t_i \) are variations of \( i \)’s strategy \( \sigma_i \). The Envelope Theorem asserts, therefore, that the gradient vector \( U_i'(t_i) \) of \( U_i \) satisfies

\[
U_i'(t_i) = V_i(t_i) \quad \text{where} \quad V_i(t_i) \equiv \frac{\partial \tilde{U}_i(t_i; \sigma)}{\partial t_i}_\sigma = I.
\]

Interchanging the order of differentiation and expectation, direct computation yields\(^{11}\)

\[
V_i(t_i) \equiv \mathcal{E} \left\{ \sum_j x_j(t) \{ u_{ij}(s_{ij}(z_j(\theta); t); \theta)\phi_i(\theta) + v_{ij}(s_{ij}(z_j(\theta); t); \theta)\} \mid t_i \right\}.
\]

\(^{10}\) Essentially, a member with inferior information need not report his type; examples are in Kennan and Wilson (1992) among many others. In subtler formulations than the one here, individuals’ overlapping information can be used to check partially the veracity of the report made by each. The extreme case of nonexclusive information is studied by Postlewaite and Schmeidler (1986).

\(^{11}\) Differentiate \( \int \int \sum_j x_j u_{ij}(s_{ij}; t_i; \tau) f(t) dF(\tau) / f_i(t_i) \) with respect to \( t_i \). In addition to the first-order necessary condition, a member’s second-order necessary condition imposes the further constraint that \( U_i''(t_i) = \partial^2 \tilde{U}_i(t_i; \sigma) / \partial t_i^2 \), or equivalently \( \partial^2 V_i(t_i) / \partial t_i^2 \), is positive semi-definite. For instance, in Myerson and Satterthwaite (1983) and Rochet (1985), \( \tilde{U}_i \) is linear in \( t_i \) so \( U_i \) must be convex. Guesnerie and Laffont (1984) and McAfee and McMillan (1988) reduce this condition to a monotonicity constraint on the allocation; such conditions are related to the condition of Bayesian monotonicity required for implementation of general social choice functions; cf. Jackson (1991) and Mookerjee and Reichelstein (1990). More generally, accurate reporting must be optimal globally. Although this constraint is binding in some applications, we ignore it here because the net effect is simply to modify the allocation obtained without this constraint to enforce monotonicity; cf. Guesnerie and Laffont (1984), Mussa and Rosen (1978), and Wilson (1992b).
Given the form (3) of the incentive-compatibility constraint as a differential equation for \( U_i \), a single boundary condition is needed for each \( i \) to ensure that \( U_i \) corresponds to its original definition. For this we use the equality

\[
\mathcal{E} \left\{ \sum_j x_j(t) u_{ij}(s_{ij}(z_j(\theta); t); \theta) - U_i(t_i) \right\} = 0 ,
\]

called the consistency constraint, which could alternatively be specified as an inequality (\( \geq \)) since its only effect is to ensure that each member’s utility is no more than the decision rule provides.

2. Derivation

Our purpose in this section is to establish a version of Myerson’s principle that the net effect of private information is to substitute “virtual” utilities for the members’ utility functions in the construction of an incentive-efficient decision rule. We first proceed generally and then derive more specific conclusions for models with simplifying features in \( \S3 \).

The maximization problem that characterizes an incentive-efficient decision rule can be posed as follows. Given contingent welfare weights, the objective is to choose a feasible decision rule that maximizes the welfare measure \( W \). Individual rationality (2) and incentive compatibility (3) impose the constraints \( U_i(t_i) \geq U^*_i(t_i) \) and \( U'_i(t_i) = V_i(t_i) \) uniformly, and in addition the utility assignments must satisfy consistency (4).

For this specification, an associated Lagrangian expression is

\[
\mathcal{L} = \mathcal{E} \left\{ \sum_i \left[ \lambda_i(t_i) U_i(t_i) + \rho_i(t_i) [U_i(t_i) - U^*_i(t_i)] + \mu_i \left[ \sum_j x_j u_{ij} - U_i(t_i) \right] \right] \right\} + \sum_i \int_{D_i} \left[ U'_i(t_i) - V_i(t_i) \right] \cdot \psi_i(t_i) \, dt_i ,
\]

where the dot (\( \cdot \)) indicates an inner product, and in the expectation some arguments are omitted for simplicity. This expression can be written out in full by using the previous formula for \( V_i \). The scalar constants \( \mu_i \), the scalar-valued functions \( \rho_i(t_i) \), and the vector-valued functions \( \psi_i(t_i) \) are Lagrange multipliers for the corresponding constraints indicating consistency (4), individual rationality (2), and incentive compatibility (3). Note that \( \rho_i \) augments the welfare weight \( \lambda_i \) sufficiently to ensure individual rationality.
We interpret the Lagrangian as the objective function for a problem in the calculus of variations in which the functions $U_i$ are to be optimized along with the decision rule. The analysis then has two parts: the first characterizes the optimal decision rule in terms of the multipliers, and the second determines the multipliers.

The Optimal Decision Rule

To characterize the optimal decision rule, we collect the relevant terms from the Lagrangian into Myerson’s virtual utilities, defined as:

$$
\bar{u}_{ij}(s; \theta) \equiv u_{ij}(s; \theta) - \left[u_{ij}(s; \theta)\phi_i(\theta) + v_{ij}(s; \theta)\right] \cdot \alpha_i(t_i),
$$

where $\alpha_i(t_i) \equiv [\mu_i f_i(t_i)]^{-1}\psi_i(t_i)$ (assuming $\mu_i > 0$). Then:

- For each project $j$ and each likely outcome $z_j$, the shares $\{s_{ij}(z_j; t)\}$ maximize $E\{\sum_i \mu_i \bar{u}_{ij}(s_{ij}; \theta) \mid z_j, t\}$ subject to $\sum_i s_{ij} \leq \bar{y}_j(z_j)$. Call this maximum $\bar{u}_j(z_j; t)$. This maximization may be subject to lower bounds on the shares, say $s_{ij} \geq s_i^*$; if so and no shares are feasible then $\bar{u}_j(z_j; t) \equiv -\infty$. The maximum $\bar{u}_j$ must be interpreted properly as a supremum when no bounds are imposed on the shares.

Note that no separate provision is made for side bets or randomization. This is because all observable events on which such contingent transfers can be based are assumed to be included as part of $z_j$; in particular, the shares $s_{ij}$ are required to be measurable with respect to the observed event $z_j$ and the reports $t$. The advantages of risk sharing usually arise from concavity of the utility functions $u_{ij}$. Here, however, we see that it is concavity of the virtual utilities $\bar{u}_{ij}$ that is the source of risk-averse behavior. If some members’ virtual utilities are convex functions of their shares, then they will absorb all income risks.

- Given an optimal sharing rule for each project, the preferred projects are those that maximize $E\{\bar{u}_j(z_j(\theta); t) \mid t\}$. Let $x(t) \equiv (x_j(t))$ be a feasible assignment of probabilities to the optimal projects.

This calculation is subtler than it appears. Ordinarily it suffices to select a single project, say $x_j(t) = 1$ for one optimal project $j = j(t)$, but if no single optimal project satisfies all the individual rationality constraints then randomization is necessary.

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12 Myerson (1981) exemplifies an alternative approach in which exogenous randomization is included by conducting the analysis in terms of the concave hulls of the virtual utilities.
The Lagrange Multipliers

We follow Mirrlees’ (1976, p. 342) method to characterize the multipliers. In the present case, the Divergence Theorem states that

$$\int_{D_i} U'_i(t_i) \cdot \psi_i(t_i) \, dt_i + \int_{D_i} U_i(t_i) \nabla \cdot \psi_i(t_i) \, dt_i = \int_{\partial D_i} U_i(t_i) [\psi_i(t_i) \cdot d\xi].$$

In the second term the divergence (trace of the Jacobian) of $\psi_i$ is $\nabla \cdot \psi_i \equiv \sum_k \partial \psi_{ik} / \partial t_{ik}$ if $t_i \equiv (t_{ik})_{k=1, \ldots, K_i}$; that is, $\nabla = (\partial / \partial t_{ik})$. In the last term, the vector differential $d\xi$ is outward-normal to the boundary $\partial D_i$, so the integrand is proportional to $U(t_i) \nu_i(t_i) \cdot \psi_i(t_i)$, where $\nu_i(t_i)$ is the unit outward-normal vector at $t_i \in \partial D_i$. The rectangular shape of $D_i$ implies that on the boundary where only the $k$-th component of $t_i$ is constrained, the integrand includes only the $k$-th component of $\psi_i(t_i)$, and with a negative or positive sign according as the component $t_{ik}$ is constrained below or above.

Applying the Divergence Theorem to the terms involving $i$’s type-contingent expected utility yields

$$\int_{D_i} \left\{ [\lambda_i(t_i) + \rho_i(t_i) - \mu_i] U_i(t_i) f_i(t_i) + U'_i(t_i) \cdot \psi_i(t_i) \right\} \, dt_i$$

$$= \int_{D_i} U_i(t_i) \left\{ [\lambda_i(t_i) + \rho_i(t_i) - \mu_i] f_i(t_i) - \nabla \cdot \psi_i(t_i) \right\} \, dt_i + \int_{\partial D_i} U_i(t_i) [\psi_i(t_i) \cdot d\xi].$$

Consequently, the right side of this equality must be zero at an optimum choice of $U_i$. That is, pointwise optimization of $U_i(t_i)$ at each type $t_i \in D_i$ requires

$$\nabla \cdot \psi_i(t_i) = [\lambda_i(t_i) + \rho_i(t_i) - \mu_i] f_i(t_i).$$

In addition, at a boundary type $t_i \in \partial D_i$ the transversality condition requires that $\nu_i(t_i) \cdot \psi_i(t_i) \leq 0$, and $= 0$ if $U_i(t_i) > U^*_i(t_i)$. In particular, the rectangular shape of $D_i$ implies that $\psi_{ik}(t_i) = 0$ on the boundary where $t_{ik}$ is bounded and $U_i(t_i) > U^*_i(t_i)$.

In principle, these conditions are sufficient to determine the variable multiplier $\psi_i$ on the domain $D_i$ given any particular specification of the constant multiplier $\mu_i$. This is obvious when $i$’s type is one-dimensional because then (5) specifies an ordinary differential equation and the transversality condition selects one of its solutions. The task is subtler when the type is multidimensional. In this case the single partial-differential equation (5) allows many solutions for the $K_i$ components of $\psi_i$. Recall, however, that $V_i$ must be the gradient of $U_i$, and therefore the Jacobian of $V_i$ must be the Hessian of $U_i$; consequently, at each $t_i$ the Jacobian of $V_i(t_i)$ must be a symmetric matrix. Thus,
one must select a solution for $\psi_i$ that, via its effect on the decision rule, yields a Jacobian of $V_i$ that is symmetric.\footnote{Methods for solving such differential equations and some solved examples in the context of nonlinear pricing are described in Wilson (1992b, Chapter 13). Recall that if utilities are linear in the type parameter then the members’ second-order conditions require that the Jacobian must also be positive semi-definite.} This assures that the vector field represented by $V_i$ can be integrated to obtain the function $U_i$.

Lastly, the consistency condition (4) determines the correct value of the constant scalar multiplier $\mu_i$. This can be stated in a more convenient form by using (5) to obtain

$$\mathcal{E} \left\{ \sum_j x_j [\lambda_i u_{ij} - \mu_i \bar{u}_{ij}] \right\} = 0. \quad (6)$$

This makes explicit the dependence on $\mu_i$, but one must remember that the decision rule $(x, s)$ also depends on $(\mu_i)$.

The next section applies these general conditions to models with special structural features.

3. Applications

We describe several general categories that illustrate structural features. Kennan and Wilson (1992) study the first in contexts of economic exchange and negotiation of legal disputes; and Cremér and McLean (1988) and McAfee and Reny (1992) study the second in the context of auctions. Guesnerie and Laffont (1984) study the third, fourth, and fifth in the context of principal-agent problems; and McAfee (1991), Myerson and Satterthwaite (1983), and Wilson (1985) among many others study these in the context of exchange.

**Common-Value Models:** If each member’s utility is independent of the type vector $t$ then $\bar{u}_{ij}(s; \theta) = u_{ij}(s; \tau)[1 - \alpha_i(t_i) \cdot \phi_i(\theta)]$. Further, if each member’s type is distributed independently of others’ types conditional on $\tau$, then $\phi_i$ depends only on $(t_i, \tau)$.

**Private-Value Models:** Private-value models assume that each member’s utility depends only on his own type, which is uninformative about the outcome $z_j$. In this case, $\bar{u}_{ij}(s; t) = u_{ij}(s; t_i)[1 - \alpha_i(t_i) \cdot \phi_i(t)] - \alpha_i(t_i)v_{ij}(s; t_i)$. Note that correlation enters via $\phi_i(t)$.

**Independent Types:** If each member’s type is distributed independently of others’ types and $\tau$ then $\phi_i(\theta) = 0$. 

**One-Dimensional Types:** Suppose that each member’s type $t_i$ is a scalar, in which case $\psi_i$ is also scalar-valued. For many of the models in the literature, $U_i^*$ is nonincreasing and $U_i$ is increasing.\textsuperscript{14} In this case, the individual rationality constraint is not binding at the upper boundary where $t_i = b_i$. Combining this with the condition (5) on the interior yields

$$
\psi_i(t_i) = \mathcal{E} \left\{ \mu_i - \lambda_i (\tilde{t}_i) - \rho_i (\tilde{t}_i) \mid \tilde{t}_i \geq t_i \right\} [1 - F_i(t_i)],
$$

where $\rho_i(t_i) = 0$ for those types $t_i > t_i^*$ above the highest type $t_i^*$ for whom the individual rationality constraint is binding. For instance, if the criterion is *ex ante* efficiency then

$$
\psi_i(t_i) = [\mu_i - \lambda_i][1 - F_i(t_i)] \quad \text{and} \quad \alpha_i(t_i) = [1 - \lambda_i/\mu_i][1 - F_i(t_i)]/f_i(t_i),
$$

for those types above $t_i^*$.

**Linear Utilities:** If the members’ utilities are similarly linear in income, say $u_{ij}(s, \theta) \equiv p(\theta) \cdot s + w_{ij}(\theta)$, then $\bar{u}_{ij}(s, \theta) = \bar{p}_i(\theta) \cdot s + \bar{w}_{ij}(\theta)$ where $\bar{p}_i = (\bar{p}_{ih})$,

$$
\bar{p}_{ih} \equiv p_h - [p_h \phi_i + \partial p_h/\partial t_i] \cdot \alpha_i \quad \text{and} \quad \bar{w}_{ij} \equiv w_{ij} - [w_{ij} \phi_i + \partial w_{ij}/\partial t_i] \cdot \alpha_i
$$

are analogous virtual representations. In this case, even if the price $p$ is constant, the virtual prices $\bar{p}_i$ affect the optimal sharing rule when the types are correlated.

Because this case occurs prominently in applications, we mention its chief simplifying feature when the criterion is *ex ante* efficiency and members’ types are independent. In this case $\phi_i \equiv 0$ and therefore $\mu_i = \mu$ for all $i$, since otherwise the Lagrangian would be unbounded. The decision rule that maximizes the group’s *ex ante* expected surplus is obtained from uniform constant welfare weights, say $\lambda_i = 1$ for all $i$; some applications also address the analogous monopoly formulation in which $\lambda_i = 0$ except for one member, say $i = 1$, for whom $\lambda_1 = 1$. To determine the single scalar multiplier $\mu$ in a way that obviates specification of the sharing rule, the members’ consistency conditions (4) or (6) can be replaced by the aggregate condition

$$
\mathcal{E} \left\{ \sum_j x_j \sum_i \left[ \lambda_i u_{ij} - \mu \bar{u}_{ij} \right] \right\} = 0 \quad \text{or} \quad \mathcal{E} \left\{ \sum_j x_j \sum_i \left[ u_{ij} - \beta_i [u_{ij} \phi_i + v_{ij}] \right] \right\} = 0,
$$

(7)

\textsuperscript{14} See Guesnerie and Laffont (1984) and Wilson (1992b, Chapter 8) for various sufficient conditions and Milgrom and Shannon (1991) for necessary and sufficient conditions.
where if (5) is interpreted as yielding \( \alpha_i(t_i; \mu_i) \) dependent on \( \mu_i \), then \( \beta_i(t_i) = \alpha_i(t_i; \infty) \).

For instance, if the types are one-dimensional as above then

\[
\beta_i(t_i) = \frac{\mu}{\mu - \lambda_i} \alpha_i(t_i; \mu) = \frac{1 - F_i(t_i)}{f_i(t_i)}
\]

for those types above \( t_i^* \). The aggregate condition (7) asserts that \( \mu \) is the ratio of the aggregates of the members’ actual and virtual expected utilities, which generally exceeds unity due to the informational rents obtained by members. And, it is determined by the condition that the decision rule obtained from \( \mu \) yields a nil aggregate virtual utility calculated using \( \alpha_i(t_i; \infty) \). \(^{15}\)

**Moral Hazard:** Moral hazard is not explicitly included in the formulation above. We describe briefly one way to include it. \(^{16}\) Allow each member’s utility function \( u_{ij} \) to depend also on all members’ private actions \( a_j \equiv (a_{ij})_{i=1,\ldots,m} \) taken if the non-null action \( j \) is selected. Also, allow that the probability distribution of the outcome is conditioned on \( a_j \) via a joint distribution of the form \( F(t)F_j(\tau | t, a_j) \). In this case, \( i \) ’s strategy is augmented by a second component: besides the strategy \( \sigma_i \) specifying his report \( \hat{t}_i \) depending on his type \( t_i \), he chooses an action strategy \( a_i \equiv (a_{ij})_{j=1,\ldots,n} \) of intended actions for each non-null project, depending on both his actual type \( t_i \) and subsequently contingent on the reports \( \hat{t} \) if these are observable. Whether or not a member’s intended actions are reported, the key feature is that the actual actions taken need not be subsequently observable (i.e., the action \( a_{ij} \) taken by \( i \) need not be measurable with respect to the observed outcome \( z_j \)); consequently, the sharing rule remains a function only of the reports and the subsequently observed outcome. A decision rule is now a triplet \( \{x, s, a\} \). Given any specified decision rule, \( i \) ’s type-dependent utility is supposed to reflect maximization with respect to both his report and his intended actions; consequently, the Envelope Theorem again implies that \( U_i(t_i) = V_i(t_i) \), where of course \( V_i \) is calculated using the specified decision rule. Thus, seemingly the only modification

\(^{15}\) See Myerson and Satterthwaite (1983) and Kennan and Wilson (1992) for other presentations of this condition. In \( \S 4 \) we present it in terms of a normalized version \( \psi_i^u \) of \( \psi_i \). In Myerson and Satterthwaite, \( \mu - 1 \) is the multiplier for a single constraint summarizing both consistency and incentive compatibility. Gresik and Satterthwaite (1989) and Milgrom (1979) are two of several studies of exchange contexts showing that informational rents vanish as the number of participants with (conditionally) independent information increases.

\(^{16}\) See Mirrlees (1976b). Using essentially this formulation, Laffont and Tirole (1986) provide a completely solved example in the context of contracting between a principal and agent; however, they also use some features peculiar to the example.
in the procedure occurs in the optimization of the decision rule, which now includes a choice of the members’ action strategies; in particular, one optimizes with respect to the virtual utilities, and each member $i$’s strategy $a_i$ must be measurable with respect to $(t_i, \hat{t})$ — or on the presumption that reports are accurate, $t$-measurable. Nevertheless, it is well known that strong assumptions are required to assure that this approach yields valid necessary conditions for the design problem that can be expressed in terms of the first-order necessary conditions for the members’ optimization of their private actions; cf. Rogerson (1985) and Jewitt (1988).

4. Auctions

An auction is the special case in which a project and its sharing rule assign transactions to the traders in a market. In this section we describe applications of the preceding results to the context of pure exchange in which there is no production ($y_j \equiv 0$) and no post-trade information ($z_j \equiv 0$) on which to condition contracts. A decision rule $(x, s)$ is represented hereafter as an assignment $(q, P)$ that specifies the commodity bundle $q_i(t)$ received and the payment $P_i(t)$ made by trader $i$ when $t$ is the list of the traders’ reported types. Similarly, represent a trader’s utility as $u_i(q, -P_i; \theta)$ on the assumption that each trader cares only about his own components of the assignment. To exclude a motive for randomization, assume that $u_i$ is smooth, increasing, and concave with derivatives that increase with $\theta$; further, the components of $\theta$ are affiliated random variables with increasing hazard rates (if they are independent) or monotone likelihood ratios (if correlated).

To simplify we assume that utilities are linear in money, say $u_i(q_i, -P_i; \theta) \equiv w_i(q_i; \theta) - P_i$. Also, each trader’s individual rationality constraint is represented by his option to forego participation in the market and thereby obtain the null trade, whose utility is normalized to zero: $U_i^+(t_i) \equiv u_i(0, 0; \theta) \equiv 0$. For the welfare criterion we use ex ante efficiency throughout: $\lambda_i(t_i) \equiv \lambda_i$.

Suppose there are $\ell$ commodities in addition to money, which is one-dimensional. For each trader $i$ the set of feasible trades of commodities is a subset $Q_i \subset \mathbb{R}^\ell$ and we assume that the set of feasible monetary payments is $\mathbb{R}$. The set of feasible assignments

\begin{itemize}
  \item[$17$] Cf. Milgrom and Weber (1982). Affiliation requires nonnegative correlation on every rectangle. Its principal implication is that the conditional expectation of any increasing function of $\theta$ is increasing in the conditioning variables. Milgrom and Shannon (1991) provide the necessary and sufficient conditions for the requisite monotonicity properties that obviate randomization.
\end{itemize}
is restricted by the aggregate feasibility conditions

\[ \sum_i q_i = 0 \quad \text{and} \quad \sum_i P_i = 0. \]

Due to these restrictions, the optimal auction design is obtained by solving an amended version of the Lagrangian problem in \( \S 2 \) in which additional multipliers \( \pi(t) \) and \( \nu(t) \) are used to include the two aggregate feasibility conditions. The necessary conditions in this case are best stated in terms of the normalized multipliers \( \rho_i^0 \equiv \rho_i / [\mu_i - \lambda_i] \) and \( \psi_i^0 \equiv \psi_i / [\mu_i - \lambda_i] \). The analog of (5) is therefore

\[ \nabla \cdot \psi_i^0(t_i) = [\rho_i^0(t_i) - 1]f_i(t_i), \quad (8) \]

together with the analogous transversality conditions. The remaining conditions can then be stated in terms of \( \beta_i(t_i) \equiv \psi_i^0(t_i)/f_i(t_i) \). In particular, \( \alpha_i(t_i) \equiv R_i \beta_i(t_i) \) where \( R_i \equiv [\mu_i - \lambda_i] / \mu_i \) is called the Ramsey number in theories of nonlinear pricing, as we explain below.

Initially we address the case that each \( Q_i = \mathbb{N}^i \). In this case, using the assumption that utilities are linear in money, the necessary conditions for finite optimal choices of the payments and the allocation are:

\[ \mu_i[1 - R_i \phi_i(t_i) \beta_i(t_i)] = \nu(t) \quad (9) \]
\[ \mathcal{E} \left\{ u_q^i(q(t); \theta) - [\mu_i R_i / \nu(t)]u_q^i(q(t); \theta) \cdot \beta_i(t_i) | t \right\} = \pi(t) / \nu(t), \quad (10) \]

where the subscripts \( q \) and \( qt \) indicate partial differentiation with respect to \( q \) and \( t_i \). Similar conditions result when utility is nonlinear in money, but they are expressed in terms of the marginal rates of substitution between money and the commodities.

**The Role of Nonlinear Pricing**

We first describe the principle that the design of an efficient auction is a version of nonlinear pricing.\(^{18}\) This principle is quite general but it suffices here to illustrate the correspondence for the special case of independent private values. That is,

\[ u^i(q; \theta) \equiv u^i(q_i; t_i) \quad \text{and} \quad F(\theta) \equiv F_1(t_1) \cdots F_m(t_m). \]

\(^{18}\) Bulow and Roberts (1989) obtain analogous results in the case that indivisible items are traded. Mirrlees (1976a) and Wilson (1992) present expositions of the theory of nonlinear pricing.
This case has the simplifying feature derived from (9) that \( \mu_i = \mu \) and \( \nu(t) = \mu \) for each contingency \( t \). Consequently, (10) can be written as

\[
u_q^i(t_i, t_i) - R_i u_q^i(t_i, t_i) \cdot \beta_i(t_i) = \pi(t)/\mu.
\] (11)

To relate these conditions to nonlinear pricing, we address a simple version of the general problem studied by Mirrlees (1976a). Consider a regulated firm that incurs the cost \( p \cdot x \) to supply a commodity bundle \( x \) to any customer in any of several markets indexed by \( i \). There are many customers in each market, each identified by his type \( t_i \), and the variety of types in market \( i \) is given by the distribution function \( F_i(t_i) \). The firm’s objective is to maximize the aggregate of consumers’ surplus subject to the constraints that its net revenue is \( r_i \) in each market \( i \). To do this the firm offers an outlay schedule \( O_i \) in market \( i \) that requires a customer to pay \( O_i(x) \) for the bundle \( x \). Customer \( t_i \) chooses the bundle \( x(t_i) \) that maximizes his net benefit \( u^i(x, t_i) - O_i(x) \) and pays \( P_i(t_i) = O_i(x(t_i)) \). As shown by Mirrlees (1976a) and Guesnerie and Laffont (1984), the firm’s problem can be posed as follows, using \( U_i(t_i) \equiv \max_x u^i(x, t_i) - O_i(x) \).

The objective is to maximize the consumers’ surplus \( \sum_i \int_{D_i} U_i(t_i) \, dF_i(t_i) \) subject to the participation constraints \( U_i(t_i) \geq 0 \), the incentive constraints \( U^i(t_i) = u^i(x(t_i), t_i) \), and the revenue constraints \( \int_{D_i} [O_i(x(t_i)) - p \cdot x(t_i)] \, dF_i(t_i) = r_i \), where \( O_i(x(t_i)) \equiv w^i(x(t_i), t_i) = U_i(t_i) \). For this maximization, the firm chooses the assignments \( U_i(t_i) \) and \( x(t_i) \) of the utility and bundle obtained by each type \( t_i \) in each market \( i \). To accomplish this one uses multipliers \( \rho_i(t_i), \psi_i(t_i) \), and \( \mu_i \) for the three constraints to form a Lagrangian objective as above.

We now indicate how solutions to this nonlinear pricing problem provide the solution to the auction design problem. In the nonlinear pricing problem, let \( x(t_i, p, r_i) \) be type \( t_i \) ‘s assigned bundle when the firm’s cost vector is \( p \) and the revenue requirement in market \( i \) is \( r_i \). This allocation satisfies the necessary condition

\[
u_q^i(x_i, t_i) - R_i(p, r_i) u_q^i(x_i, t_i) \cdot \beta_i(t_i) = p, \]

(12)

analogous to (11), where \( R_i(p, r_i) \) is the Ramsey number chosen to ensure that the revenue requirement is met. The parallel between (11) and (12) indicates that the auction design can be constructed by adjusting the cost vector \( p \) and the revenue requirements \( r_i \) in each contingency \( t \) so that \( p(t) = \pi(t)/\mu \) and \( R_i(p(t), r_i(t)) = R_i \). If the traders are identical \( ex \, ante \) (namely \( u^i, F_i \), and \( \lambda_i \) are independent of \( i \)) then the latter is a simple
one-dimensional problem because $R_i$ is independent of $i$ in both the nonlinear-pricing and auction-design problems, so it suffices to solve for a single revenue requirement $r(t)$ that is the same for all traders.

This construction has the following interpretation. In the auction there are only $m$ traders rather than many customers in several markets, and each trader $i$ has type $t_i$; moreover, feasibility requires that if the list of their reported types is $t$ then $\sum_i q_i(t) = 0$ and $\sum_i P_i(t) = 0$. For each specification $r = (r_i)$ of the revenue requirements, the commodity-balance constraint is satisfied by using the appropriate cost $p(t, r)$ in the nonlinear pricing problem. And given the cost $p(t, r)$ the money-balance constraint is met by ensuring constancy of the Ramsey numbers; that is, choose $r(t)$ to solve $R_i(p(t, r), r_i) = R_i$ for each trader $i$, which yields $p(t) = p(t, r(t))$.\(^{19}\)

Thus, to design an optimal auction one first constructs the solutions to a family of nonlinear pricing problems parameterized by the cost vector $p$ and the revenue requirement $r$, using the distribution $F$ of potential types to describe the variety of customers. When the $m$ actual traders appear at the auction one can use either of two procedures. In the direct revelation game, they report their types $t$ and are then assigned bundles $q_i(t)$ and payments $P_i(t)$ from the nonlinear pricing schedule (and possibly side payments as in fn. 19) based on the cost vector $p(t)$ and revenue requirement $r(t)$ that assure feasibility. In fact, for each trader the assigned bundle is an optimal choice from the nonlinear pricing schedule based on that cost and revenue requirement. Consequently, an alternative implementation has each trader $i$ submit a schedule $q_i(t_i; p, r_i)$ indicating his preferred choice from each possible nonlinear pricing schedule $(p, r_i)$. One then chooses the actual values of $p$ and $r$ to implement by adjusting them so that the aggregate net trade is zero. The construction is more complex in general formulations but the basic correspondence principle persists.

It is worth noting that as the number of traders increases the distribution of types appearing at the auction converges to the theoretical distribution $F$, provided each falls within one of a finite number of classes $i$ characterized by $u^i$ and $F_i$. The correct

\(^{19}\) This assures that money balance is feasible but does not necessarily construct the actual payments, which relies on the technique in Gresik and Satterthwaite (1989). The incentive conditions determine only each type’s expected payment. To complete the construction, each trader is assigned an additional side payment whose expectation is zero for each type and that depends only on others’ reported types (so as to leave the participation and incentive constraints unaltered) such that in each contingency the aggregate of these side payments offsets the sum of the outlays derived from the nonlinear pricing problem. See also McAfee (1991) and McAfee and McMillan (1988).
choices of $p$ and $r$ become perfectly predictable and cannot be affected significantly by any one trader. At any limit point, predictability implies that the Ramsey number $R_i(p, 0)$ is nil in the associated nonlinear pricing problem: the linear outlay schedule $O_i(x) = p \cdot x$ solves the pricing problem with $r_i = 0$, which is sufficient to ensure money balance in every contingency. In parallel in the auction design problem, nonmanipulability implies that the imputed costs of the incentive constraints shrink and informational rents disappear.\(^{20}\) Therefore, if the welfare weights are symmetric (to exclude monopoly power), say $\lambda_i = 1$, then at the limit $\mu = 1$, $R_i = 0$, and each trader’s transaction is determined entirely by the ordinary demand condition for his type: $u_i'(q_i, t_i) = p$. That is, each limit outcome is a Walrasian equilibrium in which the market is cleared by a fixed uniform price vector $p$ derived from the population average of the traders’ demand functions.

**A Simple Symmetric Market**

For this example and the next we use a symmetric version of the previous model, including $\lambda_i = 1$ and therefore $\mu_i = \mu$, and suppose there is a single commodity and a single type parameter ($\ell = 1$, $K_i = 1$). For this first example, suppose $u(q_i; t_i) = t_i q_i$; further, each trader is endowed with one unit of the good and can sell or buy up to one unit. Thus, $Q_i = [-1, 1]$ and $i$’s virtual utility is

$$\tilde{\alpha}(q_i; -P_i; t_i) = \tilde{t}_i q_i - P_i,$$

where $\tilde{t}_i = t_i - R \beta_i(t_i)$, $R = [\mu - 1] / \mu$, and therefore the aggregate virtual utility is $\tilde{\alpha}(q; t) = \mu \sum_i \tilde{t}_i q_i$, which is to be maximized subject to the constraint $\sum_i q_i = 0$. The optimal allocation is obtained by choosing a price $p(t)$ that provides a median for the reported distribution of the virtual types $\tilde{t}_i$ and then allocating $+1$ unit to those traders (the buyers) for whom $\tilde{t}_i > p(t)$ and $-1$ unit to those (the sellers) for whom $\tilde{t}_i < p(t)$.

This example has the property that if the number $m$ of traders is odd then the incentive constraints are not binding ($R = 0$) and the outcome is efficient. To see this suppose $P_i(t) = p(t) q_i(t)$ and observe that in this case $p(t)$ is precisely the median of the reported types, and the agent at this median does not trade. Consequently, each trader perceives that when truthful reporting would enable him to (say) buy at a favorable price, false reporting could reduce the price only by making his report the median and thereby excluding trade, or below the median and thereby being assigned to sell at a price below

\(^{20}\) Roberts and Postlewaite (1976) prove essentially this in great generality.
his true valuation. When \( m \) is even this simple property does not hold because false reporting could affect the price without eliminating the opportunity to trade. It also fails if traders are designated \textit{a priori} as either sellers or buyers. However, in the following variant we examine a more elaborate version in which the design of the payment rule is critical.

A Symmetric Market with Linear Demands

For this example we assume initially that \( Q_i = \mathcal{R} \) and each trader’s utility is \( u(q_i; t_i) = t_i q_i - [1/2]q_i^2 \). The virtual utilities are therefore
\[
\hat{u}(q_i, -P_i; t_i) = u(q_i; \hat{t}_i) - P_i \quad \text{and} \quad \hat{u}(q; t) = \mu \sum_i u(q_i; \hat{t}_i),
\]
which yields the optimal allocation
\[
q_i(t) = \hat{t}_i - p(t) \quad \text{where} \quad p(t) = \frac{1}{m} \sum_{i=1}^m \hat{t}_i.
\]

First we check that the incentive constraints are binding when the naive linear payment rule is used: \( P_i(t) = p(t)q_i(t) \). If they are not binding then \( R = 0 \) and \( \hat{t}_i = t_i \), so \( i \) obtains the net profit \( U_i(t_i; I) = [1/2] \mathcal{E} \{ [t_i - p(t)]^2 \mid t_i \} \). From this it follows that \( i \) gains by reporting a false type that differs from his true type in the direction of the difference between the mean \( \hat{t} \) of the distribution of types and his true type; e.g., if his type is high then he expects to be a net buyer and therefore prefers to decrease the price by reporting a type lower than his true type. Thus, the incentive constraint is binding if \( t_i \neq \hat{t} \).

For a full analysis it is essential to construct the correct virtual types to determine the optimal assignment. This requires the solution of (8) to obtain \( \psi_i^\circ(t_i) \), and thereby \( \beta_i(t_i) = \psi_i^\circ(t_i)/f_i(t_i) \), and then the determination of \( \mu \) and \( R \). In the present case it is clear \textit{a priori} that \( U_i(t_i) > 0 \) for both extremes \( t_i = a \) and \( t_i = b \) at the lower and upper ends of the support of the type distribution, since they are almost sure to trade at a favorable price. Consequently, the transversality condition requires that \( \psi_i^\circ(t_i) = 0 \) for these two types, and then the differential equation (8) implies that
\[
\psi_i^\circ(t_i) = \begin{cases} 
-F_i(t_i) < 0 & \text{if } a \leq t_i \leq a^* \\
1 - F_i(t_i) > 0 & \text{if } b^* \leq t_i \leq b
\end{cases}
\] (13)
where \( a^* \leq t_i \leq b^* \) is the interval for which \( U_i(t_i) = 0 \). In particular, on this interval \( \rho_i^\circ(t_i) > 0 \) and it is necessarily such that the solution to (7) and (8) provides a smooth
path between $a^*$ and $b^*$; the continuity of the path determines $\mu$. Note that $U_i(t_i) = 0$ for a type $t_i \in [a^*, b^*]$ near the mean $\bar{t}$ because when, say, $t_i > \bar{t}$ he could sustain a loss by selling at a price $p(t)$ for which $t_i > p(t) > \bar{t}$.

This example indicates how convergence to an efficient outcome is obtained as the number of traders increases. As $m$ increases and $\mu$ declines to 1, the left and right segments of $\psi_i^0$ in (13) remain invariant, but $\rho_i^0 \equiv \rho_i / [\mu - 1]$ increases. Consequently, the rising segment over the interval $[a^*, b^*]$ becomes steeper, implying that this interval shrinks and eventually vanishes. Thus, the effect of additional traders is two-fold: (1) the reduction of $\mu$ brings the virtual types closer to the actual types and thereby sets the price closer to the Walrasian price; and (2) the interval of those types of each trader who obtain zero expected gain from trade shrinks. At the limit each type’s expected utility is $[1/2] \| t_i - \bar{t} \|^2$ as calculated previously.

The characterization is essentially unaltered if traders are designated ex ante as either sellers $(Q_i = \mathbb{R}_+)$ or buyers $(Q_i = \mathbb{R}_-)$. For the sellers $\psi_i^0(t_i) = -F_i(t_i)$ and for the buyers $\psi_i^0(t_i) = 1 - F_i(t_i)$; those sellers with $t_i > a^*$ and those buyers with $t_i < b^*$ obtain $U_i(t_i) = 0$. And it is unaltered if the sets $Q_i$ are further modified to reflect trading of discrete items, as in Wilson (1985).

In fact, however, particular cases of this example, like the one before, have the property that an optimal allocation is efficient. That is, $R = 0$ and $\bar{t}_i = t_i$ independently of the number $m$ of traders. As we have seen previously, this must be due to a nonlinear payment rule. In the following, we establish this result for a more general setup with $\ell$ commodities and $K_i = \ell$ type parameters for each trader; also, $Q_i = \mathbb{R}^\ell$. However, we address only the special case that the types are uniformly distributed.

Efficiency of the Allocation

This example admits an explicit solution when the distribution of each trader’s type is uniform or Normal on a ball. Using the methods illustrated in Wilson (1992, Chapter 13) for nonlinear pricing, one can solve completely various specifications in which there are $\ell$ commodities, each trader’s type is $\ell$-dimensional, and $u_q(q_i, t_i)$ is the vector $t_i - q_i$. We address here only the case that each $t_i$ is uniformly distributed on the ball on which $z(t_i) \leq r$, where $z(t_i) \equiv [1/2] \sum_k t_{ik}^2$.

The clue is to anticipate that $U_i(t_i) = w(z(t_i))$ for some univariate function $w$.\footnote{This insight is due to Mark Armstrong (1992, personal communication).} If this is so then the Envelope Theorem’s implication that $U_i'(t_i) = \mathbb{E} \{ q_i(t) \mid t_i \}$ can be
inserted into the formula for the optimal allocation to obtain
\[ \frac{m - 1}{m}[t_i - R\beta_i(t_i)] - w'(z(t_i))\bar{t}_i = \frac{1}{m} \sum_{j\neq i} \kappa_j , \]
where \( \kappa_j = \mathcal{E}\{t_j - R\beta_j(t_j)\} \). Differentiation of this relationship, and using (8), yields the differential equation
\[ \frac{m - 1}{m}[\ell + R] - 2w''(z)z - \ell w'(z) = 0 , \]
on the domain for which \( \rho_i^7(t_i) = 0 \), so
\[ w'(z) = \frac{m - 1}{m}[1 + R/\ell] + \frac{C_0}{z^{1/2}} . \]

To determine the constant of integration \( C_0 \) we use the transversality condition, which in this case requires \( \psi^5_i(t_i)\cdot t_i = 0 \) when \( z(t_i) = r \) on the presumption as above that \( U(t_i) > 0 \) on the boundary. By symmetry around the origin of the conditional distribution of \( t_j \) given \( z(t_j) \), each \( \kappa_j = 0 \), so on the boundary the formula for the optimal allocation yields \( w'(r) = [m - 1]/m \), which determines \( C_0 \). In sum, we obtain
\[ w'(z) = \frac{m - 1}{m}[1 - R y(z)] \quad \text{and} \quad \beta_i(t_i) = y(z)\bar{t}_i , \]
where \( y(z) \equiv [1/\ell] \left[ (r/z)^{1/2} - 1 \right] . \)

These formulas apply outside an interior ball of types on which the participation constraint is binding. Its boundary \( z(t_i) = z^0 \) is identified by the condition that \( w \) is smooth across the boundary: because \( w(z) = 0 \) inside, this requires \( w'(z) = 0 \) or equivalently \( y(z) = 1/R \) and \( \bar{t}_i = 0 \) inside \( z(t_i) \leq z^0 \). Then \( w(z) \) is determined by the boundary condition that \( w(z) = 0 \) for \( z \leq z^0 \), so for \( z \geq z^0 \) (and omitting the special case \( \ell = 2 \):
\[ w(z) = C_1 + \frac{m - 1}{m} z \left[ 1 + R \frac{1 + 2y(z)}{\ell - 2} \right] , \]
\[ C_1 = -\frac{m - 1}{m} \left[ \frac{\ell + R}{\ell - 2} \right] z^0 \quad \text{and} \quad z^0 = r[R/(\ell + R)]^{2/\ell} . \]

The last step uses the aggregate form of (4) to determine \( R \) or equivalently \( \mu \). Although one can calculate (4) explicitly, the important feature is that \( w \) depends on \( m \) only via the multiplicative factor \( [m - 1]/m \) and this is also true of \( \mathcal{E}\{u(q_i(t),\bar{t}_i)\} \) for each trader \( i \). Consequently, the dependence on the number of traders drops out and the
root $R = 0$ is always a solution. Thus, the optimal allocation is efficient in each
contingency $t$.

The remarkable aspect of this conclusion is that the allocation is efficient but, except
at the limit as $m \to \infty$, it cannot be achieved with the linear payment rule assumed in
Walrasian models of markets. From d’Aspremont and Gérard-Varet (1979, Theorem 6)
we know that when $R = 0$ the nonlinear payment rule that achieves efficiency is

$$P_i(t) = -\mathcal{E} \left\{ \sum_{j \neq i} u_j(q_j(t), t_j) \mid t_i \right\} + \frac{1}{m-1} \sum_{j \neq i} \mathcal{E} \left\{ \sum_{k \neq j} u_k(q_k(t), t_k) \mid t_j \right\}.$$  

**Correlated Types and Common-Value Auctions**

The construction above assumes that the traders’ types are independent. Even when
they are correlated, possibly because they include information about a common-value
component of their utilities, the optimal auction assures that the allocation is efficient
in every contingency. The construction of the optimal payment scheme in this case
includes penalties and rewards for each trader, contingent on others’ reported types,
that are designed to ensure truthful reporting, even if the trader’s expected gain from
trade is nil. However slight they might be, correlations among the types can be exploited
to ensure truthful reporting by making the penalties and rewards sufficiently large.

This can be seen indirectly in our formulation by noting that the optimal payments
$P_i(t)$ minimize $\sum_i P_i \mu_i \xi_i(t)$ subject to $\sum_i P_i = 0$, where if $\xi_i(\theta) = 1 - R_i \phi_i(\theta) \beta_i(t_i)$
then

$$\xi_i(t) \equiv \mathcal{E} \{ \xi_i(\theta) \mid t \} = 1 - R_i \phi_i(t_i) \beta_i(t_i) \quad \text{and} \quad \phi_i(t) \equiv \mathcal{E} \{ \phi_i(\theta) \mid t \}.$$  

Note that $\phi_i$ depends nontrivially on all the types if they are correlated, even if they are
conditionally independent given $\tau$, provided the dependence on $\tau$ is nontrivial. Any
finite solution to the minimization problem requires that $\mu_i \xi_i(t) = \nu(t)$, where again
$\nu(t)$ is a multiplier for the money-balance constraint. This condition can be satisfied in
general only if $\mu_i R_i \beta_i = 0$ uniformly. This implies that each $\mu_i = \nu(t)$ and therefore
$\mu_i = \mu \equiv \max_i \lambda_i > 0$. Further, if $\lambda_i = \mu$ then $R_i = 0$; or if $\lambda_i < \mu$ then $\beta_i = 0$, which
implies $\psi_i^0 = 0$, $\rho_i^0 = 1$, and $U_i = 0$. Thus, those traders accorded inferior welfare

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22 Using *Mathematica* I have verified for small values of $\ell$ that this is the only real root.
23 This assumes $m > 1$. Singular exceptions occur but the ones I have studied for
$m = 2$ violate affiliation, and for $m > 2$ the degree of singularity escalates.
weights obtain none of the expected gains from trade, which are captured entirely by those with the maximal welfare weight. For instance, if only one trader has the maximal welfare weight, then only he profits from the auction.\textsuperscript{24}

In any case, the commodity allocation is efficient because \( \alpha_i(t_i) = R_i \beta_i(t_i) = 0 \) for every trader and therefore the virtual utility of each allocation agrees with the actual utility: the condition for an optimal allocation is

\[
\mathcal{E} \left\{ u^i_q(q_i(t); \theta) \mid t \right\} = p(t),
\]

where \( p(t) \) is a price that ensures feasibility. For instance, in the symmetric common-value case each \( u^i(q_i; \theta) = u(q_i; \tau) \) and the only efficient allocation is the null trade \( q_i(t) = 0 \), which is analogous to the no-trade theorem of Milgrom and Stokey (1982). Similarly, if we add a seller who has no information and wants simply to dispose of an inelastic supply \( S \) at a nonnegative price, then each \( q_i(t) = 0 \) or \( S/m \) as the price \( p(t) \) is negative or positive. This is true regardless of the welfare weight accorded the seller, but if the seller’s weight is greater than the others then he obtains all the gains from trade via the payments.

More realistic payment rules can be derived by imposing the participation constraint in its \textit{ex post} form; that is, each trader’s expected profit is nonnegative in every contingency \( t \). The formulation can be adapted to this modification by adjoining the \textit{ex post} participation constraint. Then the condition for optimality of the payments requires only that \( \mu_i \xi_i(t) \leq \nu(t) \), where \( \nu(t) = \max_i \mu_i \xi_i(t) \) and the difference \( \mu_i \xi_i(t) = \nu(t) - \mu_i \xi_i(t) \geq 0 \) is the Lagrange multiplier used for the \textit{ex post} participation constraint. The condition for the optimality of the allocation is then

\[
\mu_i \mathcal{E} \left\{ u^i_q(q_i(t); \theta)[\xi_i(\theta) + \zeta_i(\theta)] - R_i u^i_q(q_i(t); \theta) \beta_i(t_i) \mid t \right\} = \pi(t),
\]

where again the price \( \pi(t) \) ensures feasibility of the commodity allocation.

\textsuperscript{24} This conforms to the results obtained by Cremér and McLean (1988) who examine correlation among the bidders’ private values, and McAfee, McMillan, and Reny (1989) in the case of a common value, for an auction (of discrete items) designed to maximize the seller’s expected profit. They show that, in the limit as the bounded domain of payments is enlarged, the seller obtains all the gains from trade in every contingency \( t \). McAfee and Reny (1992) construct the payment rules, which are quite complicated. This conclusion depends on utilities being linear in money; Robert (1991) shows that it is false if utilities are strictly concave or payments are bounded. It is also false if an \textit{ex post} participation constraint is invoked.
To illustrate the implications of this modification, consider the symmetric common-value model where \( u^i(q_i; \theta) = u(q_i; \tau) \) and the types are conditionally independent and identically distributed given \( \tau \). For this model the optimality condition for the allocation can be phrased as

\[
\nu(t) \mathcal{E} \{ u^i_q | t \} - \mu_i R_i \text{Cov} \{ u^i_q, \phi(t_i; \tau) | t \} \beta_i(t_i) = \pi(t),
\]

where \( u^i_q \equiv u_q(q_i; \tau) \) and \( \text{Cov} \) indicates covariance. For example, suppose that each \( \lambda_i = 1 \), so \( \mu_i = \mu \) by symmetry, and \( u^i_q = \tau - q_i \). Also, suppose \( \theta \) has a Normal distribution for which the marginal distribution of \( \tau \) has mean \( \tau \) and variance \( \sigma^2 \), and the conditional distribution of \( t_i \) has mean \( \tau \) and variance 1. This specification implies the conditional mean and variance

\[
\mathcal{E} \{ \tau | t \} = \frac{\tau / \sigma^2 + \sum_i t_i}{1 / \sigma^2 + m} \quad \text{and} \quad \text{Var} \{ \tau | t \} = 1 / [1 / \sigma^2 + m].
\]

The optimal allocation is

\[
q_i(t) = [\mu R / \nu(t)] \text{Var} \{ \tau | t \} [\hat{\beta}(t) - \beta_i(t_i)],
\]

where \( \hat{\beta}(t) = \sum_i \beta_i(t_i) / m \) and

\[
\nu(t) = \mu + \mu R \cdot \max_i \{ [t_i - \mathcal{E} \{ \tau | t \}] - [t_i - \tau] / [\sigma^2 + 1] \} \beta_i(t_i),
\]

\[
\pi(t) = \nu(t) \mathcal{E} \{ \tau | t \} - \mu R \cdot \text{Var} \{ \tau | t \} \hat{\beta}(t).
\]

Note from the calculation of \( \nu(t) \) that typically only one trader is not constrained by the \textit{ex post} participation constraint. From the fact that \( R \to 0 \) as \( m \to \infty \) one derives \( \nu(t) \to \mu \), \( \pi(t) / \mu \to \tau \), and \( q_i(t) \to 0 \) almost surely, which is efficient. Alternatively, suppose \( \sigma \to \infty \) so that prior information becomes negligible, and the marginal variance of each type also becomes infinite. Then (8) indicates that \( \beta_i \) becomes constant (and therefore zero), so again the limit is the no-trade allocation. Thus, the complexity of this allocation rule is due mainly to the significant role of prior information compared to the traders’ private information.

5. Conclusion

The formulation in \S1 and the analysis in \S2 unify a literature that encompasses applications ranging from labor contracting to auctions. In fact, these applications are not so
disparate as they seem when encumbered by institutional detail. The aim in each case is to construct an efficient allocation subject to participation and incentive constraints, albeit in some cases with welfare weights favoring one party. Typically this involves some collective decision, some risk sharing, etc. The critical constraint in each case is incentive compatibility, which is required to induce truthful revelation of private information — or in an implementation, to ensure that the allocation can be realized by equilibrium strategies. The results conform to those obtained by different methods in studies of efficient trading procedures, and at the other extreme, also to those obtained in studies of contracting where risk sharing and imperfect monitoring of productive inputs are key features. An important advantage of this formulation is that correlation among participants’ types is no impediment to the analysis, and the results extend naturally to multidimensional types (although computations remain difficult).

The main disadvantage is heavy reliance on the formulation as a direct revelation game. The results provide clues about the possible forms of implementation as, say, auctions in the case of trading contexts, but they do not provide explicit characterizations of the efficient game form when participants’ actions are bids and offers. This is especially limiting when one envisions implementations with important dynamic features, such as a double auction with oral outcry of ask and bid prices and acceptances of offered transactions. In practice, therefore, the results are useful mainly as standards against which to compare the performance of subjects in experimental auctions. In this sense, the prospect of including risk aversion, correlation, and other realistic features is the main advantage over existing results.


