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Strategic Analysis of Auctions

Robert Wilson *

In many markets, transaction prices are determined in auctions. In the most common form, prospective buyers compete by submitting bids to a seller. Each bid is an offer to buy that states a quantity and a maximum price. The seller then allocates the available supply among those offering the highest prices exceeding the seller's asking price. The actual price paid by a successful bidder depends on a pricing rule, usually selected by the seller: two common pricing rules are that each successful bidder pays the price bid; or they all pay the same price, usually the highest rejected bid or the lowest accepted bid.

Auctions have been used for millennia, and remain the simplest and most familiar means of price determination for multilateral trading without intermediary 'market makers' such as brokers and specialists. Their trading procedures, which simply process bids and offers, are direct extensions of the usual forms of bilateral bargaining. Auctions also implement directly the demand submission procedures used in Walrasian models of markets. They therefore have prominent roles in the theory of exchange and in studies of the effects of economic institutions on the volume and terms of trade. Their allocative efficiency in many contexts ensures their continued prominence in economic theory. They are also favored in experimental designs investigating the predictive power of economic theories.

Auctions are apt subjects for applications of game theory because they present explicit trading rules that largely fix the 'rules of the game'. Moreover, they present substantive problems of strategic behavior of practical importance. They are particularly valuable as illustrations of games of incomplete information because bidders' private information is the main factor affecting strategic behavior. The simpler forms of auctions induce normal-form games that are essentially 'solved' by applying directly the basic

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equilibrium concepts of noncooperative game theory, such as the Nash equilibrium, without recourse to criteria for selecting among multiple equilibria. The common-knowledge assumption on which game theory relies is often tenable or innocuous applied to an auction.

In this chapter we describe several forms of auctions, present the formulations used in the main models, review some of the general results and empirical findings, and indicate a few applications. The aim is to acquaint readers with the contributions to a subject in which game theory has had notable success in addressing significant practical problems — and many challenging problems remain. Several other surveys of auction theory are also available, including Engelbrecht-Wiggans (1980), Rothkopf and Harstad (1992), Kagel (1991), McAfee and McMillan (1987a), Milgrom (1985, 1986, 1989), Smith (1987), and Wilson (1985b, 1987ab) as well as the collection of articles in Engelbrecht-Wiggans, Shubik, and Stark (1983), and the bibliography by Stark and Rothkopf (1979); in addition, Cassady (1967) surveys the history and practice of auctions. The origin of the subject is the seminal work by Vickrey (1961, 1962), and later the important contributions by Griesmer, Levitan, and Shubik (1967) and Ortega-Reichert (1968), who initiated formulations in terms of games with incomplete information. There is also a literature on games with complete information emphasizing multi-market and general equilibrium formulations that is not reviewed here; cf. Dubey (1982), Milgrom (1987), Peck and Shell (1992), Schmeidler (1980), Shapley and Shubik (1977), and Wilson (1978).

1. Varieties of Auctions

The diverse trading rules used in auctions share a common feature. Each player's feasible actions specify offered net trades. The trades accepted are selected by an explicit procedure, and the transaction prices are calculated from the offered trades by an explicit formula. In effect, each trader reports a demand or supply function and then prices are chosen to clear the market. Two main categories of auctions differ according to whether the process is static or dynamic. In static versions, traders submit sealed bids: each acts in ignorance of others' bids. In dynamic versions, traders observe others' bids and they can revise their bids sequentially. Repeated auctions can be further complicated by reputational effects.

Static versions allow a useful distinction between *discriminating* pricing in which trades are accepted at differing prices, usually the prices offered for the trades accepted, and *nondiscriminating* pricing, in which for identical items a single price applies to all transactions, such as the highest rejected bid.

An important example of a static auction proceeds as follows. Each seller announces a supply function; e.g., if a single seller offers q identical indivisible items at an ask price a, then the supply function is $s(p) = q\mathbf{1}_{\{p \geq a\}}$. Then each buyer $i = 1, \ldots, n$ submits a demand function $d_i(p)$: this might be piecewise constant, indicating the maximum number of items desired at each price; or in the inverse form $b_i(x)$ it indicates the maximum price offered for the x-th item. The pricing rule then selects one price p° from among the interval of 'clearing prices' that equate aggregate demand and supply; for example, the maximum (the 'first price' rule), the minimum (the 'second price' or 'highest rejected bid' rule), or the midpoint. Nondiscriminating pricing assigns this price to all transactions, namely if $d_i(p^\circ) = x_i$ then i obtains x_i items at the uniform price p° for each item, whereas purely discriminating pricing imposes the price $b_i(x)$ for the x-th item, for each $x \leq x_i$. The process is similar if a buyer solicits offers from sellers. If there are multiple sellers and multiple buyers then nondiscriminating pricing is the usual rule in static auctions, although there are important exceptions. A variant, proposed by Vickrey (1961), assigns to each bidder a clearing price that would have resulted if he had not participated.

In the simplest case, a single seller offers a single item and each buyer submits a single bid. The item is sold to the bidder submitting the highest bid at either the price he bid (discriminating) or the greater of the ask price and the second-highest bid (nondiscriminating). Many variations occur in practice; cf. Cassady (1967). The seller may impose an entry fee and need not announce the ask price in advance, the number and characteristics of the participating bidders may be uncertain, *et cetera*. If nonidentical items are offered, the seller may solicit bids for each item as well as bids for lots. The process might have a trivial dynamic element, as in the case of a 'Dutch' auction in which the seller lowers the price until some buyer accepts. This is evidently a version of discriminating pricing if players are not impatient and the seller's minimal ask price is fixed in advance, because the induced game is strategically equivalent to a static discriminating auction. Auctions in which bidders have continual opportunities to raise their bids have significant dynamic elements because the bids signal information. In an 'English' auction, a seller offers a single item and she accepts the highest bid offered above her ask price as in a static auction: the dynamic feature is that buyers can repeatedly raise their bids. The Dutch variant has the seller raising her ask price until a single buyer remains willing to buy. Possibly remaining bidders do not observe the prices at which others drop out — and indeed tracking bids can be difficult if anonymity is feasible. If such observations are precluded, then an English auction resembles a sealed-bid second-price auction, and indeed is strategically equivalent if the players are not impatient. An especially important example of a dynamic auction is a bid-ask market in which traders continually make public offers of bids (to buy) and asks (to sell) that can be accepted or withdrawn at any time.

Auctions are used mostly to exchange one or several identical items for money, but in principle auctions could be used to obtain core allocations or Walrasian equilibria of barter economies involving many goods. The familiar auctions employ invariant trading rules that are unrelated to the participants' information and preferences, and even the numbers of buyers and sellers, but we shall see later that the theory of efficient auctions finds it advantageous to adapt the trading rule to the characteristics of the participants. Conversely, implementations of demand-revelation mechanisms designed to achieve efficient allocations often resemble auctions.

2. Auctions as Games

To formulate an auction as an extensive game, in principle one first takes the trading rule as specifying an extensive form that applies to the special case that the numbers of sellers and buyers are common knowledge, as well as their characteristics (preferences, endowments, etc.) and any choices by nature. That is, the procedural steps described by the trading rule generate a list of the possible complete histories of the process, and this list matches the possible plays generated by a tree in which the order of players' moves and their feasible actions at each move are specified. The procedure also specifies the information sets of each player, consisting of minimal sets of moves that can not be distinguished and for which the feasible actions are the same; perfect recall is assumed. In practice, lacunæ in the trading rule leave gaps that are filled with specifications chosen to meet the requirements of behavioral accuracy or modeling tractability.

For example, a static auction might be modeled by a tree representing simultaneous moves by all participants: each seller selects a supply function from a feasible set, and each buyer selects a demand function. Or, if sellers first announce their supplies and ask prices before the buyers move, then two stages are required. An English auction, on the other hand, requires specification of the mechanism (such as rotation or random selection) that determines the order in which buyers obtain opportunities to raise the previous high bid, and their opportunities to observe others' bids. For tractability a reduced form of the tree may be used, as in the approximation that has the seller raising her asking price.

If all information is common knowledge, this extensive form becomes an extensive game by adding specifications of the probabilities of nature's moves and the players' payoffs for each play. The players' payoffs are determined by applying their preferences to the allocation determined by the trading rule; that is, the trading rule, including its pricing rule, assigns to each play an allocation that indicates for each player the (possibly random) transfers of goods and money obtained. For example, in a single-item discriminating auction, a buyer who assigns a value v to the item may obtain the net payoff $v - p$ if he receives the item after bidding p, and zero otherwise.

In practice, however, participants' preferences are rarely common knowledge, and indeed a major motivation for using auctions is to elicit revelation of preferences so that maximal gains from trade can be realized. Thus, the actual extensive game to be studied derives from a larger extensive form in which initially nature chooses an assignment of 'types' (e.g., preferences or other private information) for the players and possibly also the set of participating players. In this larger form, each player's information sets are the unions of the corresponding information sets in the various versions of the smaller extensive form that he can not distinguish based on his private information about nature's initial choice. This larger extensive form becomes an extensive game by specifying a probability distribution of nature's choices of assignments. Thus, the induced extensive game has incomplete information.

For example, for a static single-item auction a bidder might learn those features that are common knowledge, such as the number of bidders, as well as private information represented by the valuation v he assigns to the item. In this case, his possible pure strategies are functions $b(v)$ that assign a bid to each contingency v. If the pricing is discriminating (and ignoring ties) then his expected payoff (absent risk aversion) from such a strategy is $\mathcal{E}\{[v - b(v)]\mathbf{1}_{\{b(v)>B\}}\}$ where B is the maximum of the others' bids and the ask price. Note that B is a random variable even if the other bidders' strategies are specified, since their valuations are not known. The second-price rule, on the other hand, yields the payoff $\mathcal{E}\{[v - B]\mathbf{1}_{\{b(v)>B\}}\}$ because the winning bidder pays the highest rejected price, which is B. Note that the pricing rule affects the extensive game and its normal form only via the expected payoffs assigned to strategy combinations of the players.

Static auctions with simultaneous moves conform exactly to their normal-form representation so the usual equilibrium concept is the Nash equilibrium. It is desirable, however, to enforce perfection to obtain the constraint $b(v) \leq v$ on bids with no chance of succeeding, and one focuses naturally on equilibria that preserve symmetries among the players. Dynamic elements require sequential equilibria. For example, if the seller sets an ask price first, then a Nash equilibrium might allow the seller to be deterred from setting a high ask price by expectations of lower bids or fewer bidders; or in a Dutch auction a Nash equilibrium allows bidders to expect the seller to withdraw the item before the price drops to her valuation. The role of equilibrium selection criteria in truly dynamic auctions, such as bid-ask markets, has not been studied.

3. Static Single-Item Symmetric Auctions

In this section we review a portion of the basic theory of static auctions in which a seller offers a single item and the bidders are symmetric. Our aim is indicate the formulation and methods used, because they are indicative of the approach taken in more elaborate problems. We present Milgrom and Weber's (1982a) characterization of the symmetric Nash equilibrium. The number n of bidders, the seller's ask price a, and the probability distribution of bidders' private information are assumed to be common knowledge. All parties are assumed to be risk-neutral.

The model supposes that nature assigns each bidder i a pair (x_i, v_i) of numbers. These have the following roles: bidder i observes the real-valued 'signal' x_i before submitting his bid and if he wins the item at price p then his payoff is $v_i - p$. In practice, x_i is interpreted as bidder i's sample observation, from which he constructs an initial estimate $\mathcal{E}\{v_i \mid x_i\}$ of his subsequent valuation v_i of the item, which may be observed only after the auction. Let $z = (x_i, v_i)_{i=1,..,n}$ indicate nature's choice and use $F(z)$ to denote its joint distribution function, which we assume has an associated density $f(z)$ on a support that is a rectangular cell $Z = \{z \mid \underline{z}\mathbf{1} \leq z \leq \overline{z}\mathbf{1}\}$. Let $z \vee z'$ and $z \wedge z'$ be the elements of Z that are the component-wise maximum and minimum of z and z'. And, let $x^1, \ldots, x^n$ be the components of x arranged in nonincreasing order. The conditional distribution function of x^2 given $x^1 = s$ is denoted by $\hat{F}(\cdot \mid s)$ and its density, by $\hat{f}(\cdot \mid s)$.

By symmetry of the bidders we mean that F is invariant under permutations of the bidders. In particular, the conditional distribution of v_i given $x_1, \ldots, x_n$ is invariant under permutations of those bidders $j \neq i$. A further technical assumption is called *affiliation*:

$$(\forall z, z' \in Z) \qquad f(z \vee z')f(z \wedge z') \geq f(z)f(z')\,. \tag{1}$$

This assumption states essentially that on every subcell of Z the components of z are nonnegatively correlated random variables. Its useful consequence is that the conditional expectation of a nondecreasing function of z given that z lies in a subcell, is a nondecreasing function of the boundaries of that subcell. Affiliation implies also that (x^1, x^2) has an affiliated density and that $\hat{F}$ has the monotone likelihood ratio property (MLRP); further, $\hat{f}(t \mid s)/\hat{F}(t \mid s)$ is a nondecreasing function of s.

Symmetry implies that the function

$$v(s,t) = \mathcal{E}\{\, v_i \mid x_i = x^1 = s \;\&\; x^2 = t \,\} \tag{2}$$

is well defined (i.e., it does not depend on i nor on the $j \neq i$ for which $x_j = x^2$). Moreover, affiliation implies that it is a nondecreasing function. Let $v(\underline{x}, \underline{x}) = \underline{v}$, where $\underline{z} = (\underline{x}, \underline{v})$. The central role of this function in the analysis is easily anticipated. If the symmetric equilibrium strategy makes each bidder's bid an increasing function of his signal, then the one, say i, observing the highest signal $x^1 = s$ will win and obtain the conditional expected payoff

$$V_n(s) \equiv \mathcal{E}\{\, v_i \mid x_i = x^1 = s \;\&\; x^2 < s \,\} = \mathcal{E}\{\, v(s, x^2) \mid x^2 < s \,\}\,, \tag{3}$$

given that he wins after observing the signal s, gross of the price he pays. Thus, $V_n(x_i)$ as well as $v(x_i, x_i)$ are upper bounds on the profitable bids that bidder i can submit after observing the signal x_i.

THEOREM 1 (Milgrom and Weber): Assume symmetry and affiliation, and suppose that $a \leq \underline{v}$. Then the symmetric equilibrium strategy σ in a discriminating (first-price) auction prescribes the bid

$$\begin{aligned} \sigma(x) &= [\underline{v}\theta(\underline{x}) + \int_{\underline{x}}^{x} v(t,t)\, d\theta(t)]/\theta(x) \\ &= v(x,x) - \int_{\underline{x}}^{x} \theta(t)\, dv(t,t)/\theta(x)\,, \end{aligned} \tag{4}$$

if the bidder's observed signal is x, where

$$\theta(x) = \exp\left\{ \int_{\underline{v}}^{x} \frac{\hat{f}(t \mid t)}{\hat{F}(t \mid t)}\, dt \right\}. \tag{5}$$

In a nondiscriminating (second-price) auction, $\sigma(x) = v(x,x)$.

SKETCH OF PROOF: In a first-price auction, if the strategy σ is increasing and differentiable with an inverse function X, then the optimal bid b must maximize the expected profit

$$\int_{\underline{x}}^{X(b)} [v(x,t) - b]\, d\hat{F}(t \mid x)\,. \tag{6}$$

It must therefore satisfy the necessary condition

$$\begin{aligned} 0 &= -\hat{F}(X(b) \mid x) + [v(x, X(b)) - b]\hat{f}(X(b) \mid x)X'(b)\,, \\ \text{so} \quad 0 &= \left\{ -\sigma'(x)\hat{F}(x \mid x)/\hat{f}(x \mid x) + [v(x,x) - \sigma(x)]\right\} \frac{\hat{f}(x \mid x)}{\sigma'(x)}\,, \end{aligned} \tag{7}$$

where the second equality uses the equilibrium condition that the optimal bid must be $b = \sigma(x)$, and $X'(b) = 1/\sigma'(x)$. It is also necessary that $\sigma(x)$ does not exceed $v(x,x)$ (otherwise winning is unprofitable), and that $\sigma(\underline{x}) \geq \underline{v}$ (otherwise a larger bid would be profitable when $x = \underline{x}$), and therefore this differential equation is subject to the boundary condition that $\sigma(\underline{x}) = \underline{v}$. The formula in the theorem simply states the solution of the differential equation subject to the boundary condition. Verification that this solution is indeed increasing is obtained by recalling that $v(t,t)$ is an increasing function of t and noting that as x increases the weighting function $\theta(t)/\theta(x)$ puts greater weight on higher values of t. The second version of the necessary condition implies that the expression in curly brackets is zero at the bid $b = \sigma(x)$, and since $\hat{f}(\hat{x} \mid x)/\hat{F}(\hat{x} \mid x)$ is nondecreasing in x as noted earlier, for a bid $\hat{b} = \sigma(\hat{x})$ it would be nonnegative if $\hat{x} < x$ and nonpositive if $\hat{x} > x$; thus the expected profit is a unimodal function of the bid and it follows that the necessary condition is also sufficient. The assumed differentiability of the strategy is innocuous if it is continuous since affiliation is preserved under monotone transformations of the bidders' observations x_i. Consequently, the remainder of the proof consists of showing that in general the strategy must be continuous on each of several disjoint intervals, in each of which it is common knowledge among the bidders that all observations lie in that interval. This last step also invokes affiliation to show that the domains of continuity are intervals.[1] The argument is analogous for a second-price auction except that the preferred bid is the maximum profitable one, $\sigma(x) = v(x,x)$, since his bid does not affect the price a bidder pays.

The assumption that the distribution F has a density is crucial to the proof because

[1] Milgrom and Weber (1982a, fn. 21) mention an example with two domains; at their common boundary the strategy is discontinuous.

it assures that the probability of tied bids is zero.[2] If the distribution F is not symmetric then generally one obtains a system of interrelated differential equations that characterize the bidders' strategies.

This theorem allows various extensions. For example, if the seller's ask price is $a > \underline{v}$ and $x(a)$ solves $a = v(x(a), x(a))$ then

$$\sigma(x) = \min \left\{ v(x, x), \frac{a\theta(x(a)) + \int_{x(a)}^{x} v(t,t)\, d\theta(t)}{\theta(x)} \right\} . \tag{8}$$

In this form the theorem allows a random number of *active* bidders submitting bids exceeding the ask price. That is, the effect of an ask price (or bid preparation costs) is to attract a number of bidders that is affiliated with the bidders' signals; thus high participation is associated with high valuations for participants.

The Independent Private-Values Model

If each bidder observes directly his valuation, namely the support of F is restricted to the domain where $(\forall i) \quad x_i \equiv v_i$, then $v(s,t) = s$. One possible source of correlation among the bidders' valuations is that, even though the bidders' valuations are independently and identically distributed, the bidders are unsure about the parameters of this distribution. If their valuations are actually independent, say $F(z) = \prod_i G(x_i)$, then the bidders are said to have *independent private values*. For this model, $\theta(x) = \hat{F}(x) = G(x)^{n-1}$ is just the distribution of the maximum of the others' valuations; hence

$$\sigma(x) = \min\{ x, \mathcal{E}\{ \max[a, x^2] \mid x^1 = x \} \} \tag{9}$$

in a first-price auction. In a second-price auction, $\sigma(x) = x$, which is actually a dominant strategy. The seller's expected revenue is therefore the expectation of $\max\{ a, x^2 \} \mathbf{1}_{\{ x^1 \geq a \}}$ for either pricing rule — a result often called the *revenue equivalence* theorem. This result applies also if the number of bidders is independently distributed; for example, if each participating bidder assigns the Poisson distribution $q_m = e^{-\lambda}\lambda^m/m!$ to the number $m = n - 1$ of other participating bidders, then $\theta(x) = e^{\lambda F(x)}$.[3] It also illustrates the general feature that the seller's ask price a can be regarded as another bid; for example,

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[2] Milgrom (1978, p. 56) and Milgrom and Weber (1985, fn. 9) mention asymmetric examples of auctions with no equilibria. In one, each bidder knows which of two possible valuations he has and these are independently but *not* identically distributed. An equilibrium must entail a positive probability of tied bids, and yet if ties are resolved by a coin flip then each bidder's best response must avoid ties.

[3] More generally, if k identical items are offered and each bidder demands at most

if the seller has an independent private valuation $v_0 \geq 0$ then this plays the role of an extra bid, although presumably it has a different distribution. If the seller can commit beforehand to an announced ask price, however, then she prefers to set $a > v_0$. For example, if the bidders' valuations are uniformly distributed on the unit interval, namely $G(x) = x$, then their symmetric strategy and the seller's expected revenue are

$$\sigma(x) = \min\{x, \tfrac{n-1}{n}x + \tfrac{1}{n}a^n/x^{n-1}\} \quad \text{and} \quad R_n(a) = \tfrac{n-1}{n+1} + a^n[1 - \tfrac{2n}{n+1}a], \tag{10}$$

from which it follows that for every n the seller's optimal ask price is $a = \frac{1}{2}[1 + v_0]$.

The Common-Value Model

In a situation of practical importance the bidders' valuations are identical but not observed directly before the auction. In the associated *common value* model, $(\forall i) \quad v_i = v$, and conditional on this common value v their samples x_i are independently and identically distributed. In this case, an optimal bid must be less than the conditional expectation of the value given that the bidder's sample is the maximum of all the bidders' samples, since this is the circumstance in which the bid is anticipated to win.

For example, suppose that the marginal distribution of the common value has the Pareto distribution $F(v) = 1 - v^{-\alpha}$ for $v \geq 1$ and $\alpha > 2$, so that $\mathcal{E}\{v\} = \alpha/[\alpha - 1]$. If the conditional distribution of each sample is $G(x_i \mid v) = [x_i/v]^\beta$ for $0 \leq x_i \leq v$ then the conditional distribution of the value given that an observed sample x is the maximum of n samples is

$$V_n(x) = \frac{\alpha + n\beta}{\alpha + n\beta - 1} \max\{1, x\}. \tag{11}$$

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one, then the unique symmetric equilibrium strategy (ignoring the ask price a) for discriminating pricing is $\sigma(x) = \mathcal{E}\{x^{k+1} \mid x^{k+1} < x\}$, and the seller's expected revenue is $k\mathcal{E}\{x^{k+1}\}$ for either pricing rule. Milgrom and Weber (1982) and Weber (1983) demonstrate revenue equivalence whenever the k bidders with the highest valuations receive the items and the bidder with the lowest valuation gets a payoff of zero. This is true even if the items are auctioned sequentially, in which case the successive sale prices have the Martingale property with the unconditional mean $\mathcal{E}\{x^{k+1}\}$. Harstad, Kagel, and Levin (1990) demonstrate revenue equivalence among five auctions: the two pricing rules combined with known or unknown numbers (with a symmetric distribution of numbers) of bidders, plus the English auction. For example, a bidder's strategy in the symmetric equilibrium of a first-price auction is the expectation of what he would bid knowing there are n bidders, conditional on winning; that is, each bid $\sigma_n(x)$ with n bidders is weighted by $w_n(x)$, which is the posterior probability of n given that x is the largest among the bidders' valuations.

The symmetric equilibrium strategy in this case, assuming that the seller's ask price is not binding and using $B = [n-1]\beta$, is

$$\sigma(x) = \frac{B + [\max\{1, x\}]^{-B-1}}{B+1} V_n(x)\,. \tag{12}$$

Ex ante, each bidder's and the seller's expected profit are

$$\frac{\beta}{[\alpha + B][\alpha + B + \beta - 1]}\mathcal{E}\{v\} \quad \text{and} \quad \left[1 - \frac{B+\beta}{[\alpha + B][\alpha + B + \beta - 1]}\right]\mathcal{E}\{v\}\,. \tag{13}$$

To take another example of the common-value model, suppose that the marginal distribution of the value is a Gamma distribution with mean m/k and variance m/k^2, and the conditional distribution of a sample is the Weibull distribution $G(x_i \mid v) = e^{vy(x_i)}$ where $y(x_i) = -x_i^{-\beta}$. Then the symmetric equilibrium bidding strategy is

$$\sigma(x) = \frac{m+2}{m+1+\frac{n}{n-1}}V_n(x) \qquad \text{where} \qquad V_n(x) = \frac{m+1}{k - ny(x)}\,. \tag{14}$$

In practice, bidding strategies are often constructed on the assumption that the marginal distribution of the common value has a large variance. For example, suppose that each estimate x_i has a Normal conditional distribution with mean v and variance s_1^2, and that the marginal distribution of v has a Normal distribution with variance s_0^2. If $a = -\infty$ then the limit of the symmetric equilibrium bidding strategy as $s_0 \to \infty$ is $\sigma(x) = x - \alpha_n s_1$, where

$$\alpha_n = \frac{\int_{-\infty}^{\infty} \xi^2 \, dN(\xi)^n}{\int_{-\infty}^{\infty} \xi \, dN(\xi)^n}\,, \tag{15}$$

using the standard Normal distribution function N with mean 0 and variance 1; that is, α_n is the ratio of the second to the first moment of the distribution of the maximum of n standard Normal variables.[4]

For the Lognormal distribution, suppose the conditional distribution of $\ln(x_i)$ has a Normal distribution with mean $\ln(v)$ and variance s_1^2 and the marginal distribution of $\ln(v)$ has a Normal distribution with variance s_0^2. If $a \leq 0$ then the limit of the bidding strategy as $s_0 \to \infty$ is $\sigma(x) = \beta_n(s_1)x$, where

$$\beta_n(s) = \frac{\int_{-\infty}^{\infty}[s+\xi]e^{-s\xi} \, dN(\xi)^n}{\int_{-\infty}^{\infty}[s+\xi] \, dN(\xi)^n}\,. \tag{16}$$

[4] An erroneous statement of this result in Thiel (1988) is corrected by Levin and Smith (1991), who show also that if $s_0 \equiv \infty$ then additional equilibria exist with strategies having an additional nonlinear term.

In this case, $\mathcal{E}\{v \mid x_i\} = x_i B(s_1)$, where $B(s) = e^{.5s^2}$, so it is useful to correct for bias by taking the estimate to be $x_i' = x_i B(s_1)$ and the strategy to be $\sigma(x') = [\beta_n(s_1)/B(s_1)]x'$. Table 1 tabulates a few values of α_n and $\beta_n(s)/B(s)$. Notice that as n increases, α_n first decreases due to increasing competition, and then increases due to the decline in the expected value of the item conditional on winning. That is, the supposition that x is the maximum of n unbiased estimates of v implies that x is biased by an amount that increases with n. A similar effect can be seen in the behavior of $\beta_n(s)/B(s)$.

A model with wide applicability assumes that a bidder's valuation of the item has the form $p_i v$ where p_i is a private factor specific to bidder i and v is a common factor. Before bidding, each bidder i observes (p_i, x_i), where x_i is interpreted as an estimate of the common value v. Conditional on unobserved parameters (p, v), the bidders' observations are independent, and the private and common factor components are independent. Wilson (1981) studies such a model adapted to bidding for oil leases. Assume that $\ln(p_i)$ and $\ln(x_i)$ have conditional distributions that are Normal with means $\ln(p)$ and $\ln(v)$, and variances t^2 and s^2, respectively. If the marginal variances of the unobserved parameters are infinite, then the symmetric equilibrium strategy is linear of the form $\sigma(p_i, x_i) = \gamma_n p_i \mathcal{E}\{v \mid x_i\}$. In this case the variance of the natural logarithm of the bids is $t^2 + s^2$, which for auctions of leases on the U.S. Outer Continental Shelf is usually about 1.0. This model allows the further interpretation that there is variance also in the bid factors used by the bidders (included in t^2) due perhaps to differences in the models and methods used by the bidders to prepare their bids. Table 2 displays the bid factor γ_n and the percentage expected profit of the winning bidder for several cases. These conform roughly to the one-third and one-quarter maxims used in the oil industry (Levinson (1987)): bid a third, profits average a quarter. These two fractions add to less than one because winning indicates that the estimate is biased too high.

A key feature of the equilibrium strategies identified by Theorem 1 is that each bidder takes account of the information that would be revealed by the event that his bid wins. That is, in the symmetric case, winning reveals that the bidder's sample is the maximum among those observed by all bidders. Even if each sample is an unbiased estimate of the item's value, it is a biased estimate conditional on the event that the bid wins. This feature is inconsequential if bidders directly observe their valuations, but it crucially affects the expected profitability of winning in a common-value model. Failure to take account of estimating bias conditional on winning has been called the ‘winner’s curse’; cf. Thaler (1988). In §8 we report some of the experimental evidence on its prevalence.

The following subsections provide a sampling of further results regarding static single-item auctions. The first examines the effect of increasing competition in a symmetric common-value auction.

Auctions with Many Bidders

The main results are due to Milgrom (1979ab), Wang (1991a), and Wilson (1977), who study the case of an unobserved common value v and signals that are conditionally independent and identically distributed given v. Wang assumes a discrete distribution for the signals, whereas the others assume the signals’ conditional distribution has a positive density $f(\cdot \mid v)$. Consider a sequence of symmetric auctions with discriminating pricing, all with the same common value, in which the n-th auction has bidders $i = 1, \ldots, n$ who observe the signals $x_1, \ldots, x_n$. Say that the sequence of signals is an extremal-consistent estimator of v if there exists a sequence of functions g_n such that $g_n(\max_{i \leq n} x_i)$ converges in probability to v as $n \to \infty$. Milgrom shows that the signal sequence is an extremal-consistent estimator of the common value if and only if $v < v'$ implies $\inf_x \{ f(x \mid v) / f(x \mid v') \} = 0$.

THEOREM 2 (Milgrom): For a sequence of discriminating (first-price) symmetric auctions with increasing numbers of bidders, the winning bid converges in probability to the common value if and only if the signal sequence is an extremal-consistent estimator of the common value.

Thus, whenever a consistent estimator of the common value can be based on the maximum signal, the winning bid is one such estimator, using $g_n \equiv \sigma_n$ where σ_n is the symmetric equilibrium strategy when there are n bidders.

Allowing risk aversion as well, Milgrom (1981) demonstrates comparable results for auctions with nondiscriminating pricing. The dependence on equilibrium strategies is relaxed by Levin and Harstad (1992) using the more restrictive model in Wilson (1977); e.g., the support of the bidders’ signals moves monotonely as the value changes. They show that convergence obtains if bidders’ strategies are restricted only by single-iteration elimination of dominated strategies: each bidder uses a strategy that is undominated if other bidders use undominated strategies.

An extremal-consistent estimator exists for most of the familiar distributions, such as the Normal or Lognormal. But there are important exceptions, such as the exponential distribution with mean v, for which the conditional distribution of the maximum bid is nondegenerate in the limit, although the limit distribution does have v as its mean. There seem to be no general results on the rate of convergence, but convergence is of order $1/n$ in examples — although these all have the property that this is the rate of convergence of extremal-consistent estimators.

Matthews (1984) studies the special case of the common-value model in which the conditional distribution of a bidder's signal is $F_i(x_i \mid v) = [x_i/v]^{m_i}$; that is, the signal x_i is the maximum of m_i samples uniformly distributed on the interval $[0, v]$. However, the formulation is enriched to allow that each bidder chooses both the number $m_i \in \Re_+$ of samples, at a cost $c(m_i)$, and his bid depending on the signal x_i observed. In a symmetric pure-strategy equilibrium, of course, all bidders choose the same number m of samples. Assume that c is convex and increasing, with $c(0) = 0$. Matthews establishes that as the number n of bidders increases, $m \to 0$ but the total number nm of samples purchased is bounded and bounded away from zero. Moreover, *ex ante* the expectation of the difference between the common value and the sum of the maximum bid and the sampling costs $nc(m)$ of *all* the bidders converges to zero.[5] Thus, pure-strategy equilibria necessarily entail limits on aggregate expenditures for information, and the seller expects to reimburse these expenditures. Indeed, if $c(m) = \hat{c}m$ then the maximum bid converges in probability to the common value only if $\hat{c} = 0$.

Superior Information

The familiar auction rules treat the bidders symmetrically. Consequently, the principle asymmetries among bidders are due to differences in payoffs and differences in information. Here we describe briefly some of the features that occur when some bidders have superior information.

The basic result about the effect of superior information is due to Milgrom (1979a) and Milgrom and Weber (1982b). Consider a static auction of a single item with risk-neutral bidders, and as in §3, Theorem 1, let x_i and v_i be the signal and valuation of bidder i.

THEOREM 3 (Milgrom): In an equilibrium of a first-price (discriminating) auction, a bidder i's expected payoff is zero if there is another bidder j whose information is superior (x_j reveals x_i) and whose valuation is never less ($v_j \geq v_i$). In a second-price

[5] Harstad (1990) shows that with equilibrium strategies this expectation is exactly zero.

(nondiscriminating) auction, a bidder i's equilibrium expected payoff is zero if there is another bidder j such that i's signal is a garbling of j's, j's valuation is never less, and j's bids have positive probabilities of winning in equilibrium.

Sketch of Proof: In a first-price auction, given x_j and therefore x_i, if i's strategy allows a positive expected payoff for a bid b, then j profits by bidding slightly more than b whenever he would have bid the same or less. In a second-price auction, j's (serious) bid is a conditional expectation of v_j that is independent of i's strategy (because i's bid reveals no additional information) and that implies non-positive expected payoff for j, and hence also i because i must pay at least j's bid to win.

To examine the implications of this result, consider a common-value model in which pricing is discriminating, the seller's ask price is a, and there are $m+n$ bidders. Suppose that m bidders know the common value v and n know only its probability distribution F, which has a positive density on an interval support. If $m > 1$ and $v > a$ then in any equilibrium all of the informed players bid v; consequently, an uninformed player expects to win the item only by paying more than v and incurring a loss. An equilibrium therefore requires that the uninformed players have no chance to win and obtain zero payoffs. If $m = 1$, then the informed player's unique equilibrium strategy is to bid

$$\sigma(x) = \max[a, \mathcal{E}\{v \mid v \leq x\}] \tag{17}$$

when he observes $v = x > a$. The uninformed players use mixed strategies such that the maximum b of their n bids has the distribution function

$$H(b) = \max\{F(v(a)), F(\sigma^{-1}(b))\} \tag{18}$$

if $b \geq a$, where $a = \mathcal{E}\{v \mid v \leq v(a)\}$ defines $v(a)$. That is, the uninformed players' mixed strategies replicate the distribution of the informed player's bids on the support where $\sigma(x) > a$; in addition, there may be a probability $F(v(a))$ of submitting bids sure to lose (or not bidding).[6] If the equilibrium is symmetric then each uninformed player uses the distribution function $H(b)^{1/n}$. The informed player's strategy assures that each uninformed player's expected payoff is zero; the expected payoff of the informed player

[6] This feature can be proved directly using the methods of distributional strategies in §4; cf. Engelbrecht-Wiggans, Milgrom, and Weber (1983) and Milgrom and Weber (1985).

is

$$\int_a^\infty [v - \sigma(v)]H(\sigma(v))\,dF(v) = \left\{[1 - F(v(a))][v(a) - a] + \int_{-\infty}^a F(v)\,dv\right\} F(v(a)) + \int_{v(a)}^\infty [1 - F(v)]F(v)\,dv\,. \tag{19}$$

Examples: (1) If $F(v) = v$ then $\sigma(x) = \max[a, x/2]$ if $x > a$, $v(a) = 2a$, and $H(b) = \max[2a, 2b]$; if $a = 0$ then the informed player's expected payoff is $1/6$. (2) If $a = -\infty$ and v has the Normal distribution $F(v) = N([v - m]/s)$ then $\sigma(x) = m - sN'(\xi)/N(\xi)$ where $\xi = [x - m]/s$. (3) If $a = 0$ and $F(v) = N([\ln(v) - m]/s)$ then $\sigma(x) = \mu N(\xi - s)/N(\xi)$, where $\mu = \exp(m + .5s^2)$ and $\xi = [\ln(x) - m]/s$.

These results extend to the case that the uninformed bidders value the item less. Suppose they all assign value $u(v) \leq v$, and for simplicity assume that $a \leq \underline{v} = u(\underline{v})$ where $F(\underline{v}) = 0$. Then,

$$\sigma(x) = \int_{\underline{v}}^x u(v)\,dF(v)/F(x) \qquad \text{and} \qquad H(\sigma(x)) = \exp\left\{-\int_x^\infty \left[\frac{u(v) - \sigma(v)}{v - \sigma(v)}\right] \frac{dv}{v}\right\}. \tag{20}$$

Milgrom and Weber (1982b) show that these results imply several conclusions about the participants' incentives to acquire or reveal information. Assume that the informed bidder's valuation is actually the conditional expectation $v = \mathcal{E}\{V \mid X\}$ of the ultimate common value V, given an observation X. Assume also that $a = -\infty$ and that the value is the same for all bidders.

- The bidder with superior information gains by acquiring additional information, and more so if this is done overtly; i.e., the uninformed bidders know he acquires this information.
- An uninformed bidder gains by acquiring some of the informed bidder's information, provided this is done covertly.
- The seller gains (in expectation) by publicizing any part of the informed bidder's information, or any information that is jointly affiliated with *both* the informed bidder's observation X and the common value V.[7]

Alternatively, suppose one bidder, the 'insider', knows precisely the common value v and the n other bidders obtain informative signals x_i about this common value. In

[7] Milgrom and Weber note that joint affiliation of the triplet is necessary; otherwise the seller's information can be 'complementary' to the informed bidder's information and increase his expected profit.

this case, if the insider's strategy is ρ then the appropriate extension of Theorem 1 to characterize the symmetric equilibrium strategies of those bidders other than the insider uses the revised function

$$v(s,t) = \mathcal{E}\{\, v \mid x_i = x^1 = s \;\&\; x^2 = t \;\&\; \rho(v) < \sigma(s)\} \,. \tag{21}$$

We illustrate with an example in Wilson (1975). Suppose the common value has a uniform distribution on the unit interval and the estimates x_i are uniformly distributed between zero and $2v$. The symmetric equilibrium in this case has $\rho(v) = \alpha v$ and $\sigma(x) = \min\{\alpha, \beta x\}$. Table 3 tabulates α and β when there are $n = 1$ or 2 imperfectly informed bidders. Generally, $\alpha = n/[n+1]$.

Observe that if $n = 0$ [or 1] then an additional uninformed bidder, who would otherwise submit a bid of zero, is willing to pay at most .0244 [or .0178] to acquire a signal (thus making $n = 1$ [or 2]) and submit a positive bid.

If there are additional bidders with no private information, then their best strategy is to bid zero, or not to bid, since otherwise their expected profit is negative. If there were two perfectly informed bidders then they would each bid the value v (i.e., $\alpha = 1$) and then all imperfectly informed bidders prefer not to bid any positive amount.

Asymmetric Payoffs

Strongly asymmetric outcomes can also occur if the bidders have identical information but one values the item less. Milgrom (1979a) gives an example in which two bidders assign valuations $v_1 > v_2 > a$ that are common knowledge: all equilibria have the form that 1 uses a pure strategy $b_1 \in [v_2, v_1]$ and 2 uses a mixed strategy H such that b_1 is optimal against H and 1 surely wins. Bikhchandani (1990) provides an example of repeated nondiscriminating common-value auctions in which there is a chance that one of the two bidders, say 1, consistently values the items more. Because this feature accentuates the adverse selection encountered by the winning bidder, in the equilibrium bidder 1 surely wins every auction.

Maskin and Riley (1983) provide an example indicating that the seller's choice of the auction rules can be affected by asymmetries. They consider an auction with two bidders having independent private valuations that are uniformly distributed on two different intervals. Represent these intervals as the interval from zero to $\frac{1}{1-\alpha}$ for bidder 1 and from zero to $\frac{1}{1+\alpha}$ for bidder 2. The equilibrium strategies for a first-price auction in this case are

$$X_1(b) = \frac{2b}{1-\alpha(2b)^2} \qquad \text{and} \qquad X_2(b) = \frac{2b}{1+\alpha(2b)^2} \tag{22}$$

for $b \leq .5$, indicating the valuations at which the bidders would submit the same bid b. Note that bidder 1, whose valuation is perceived by bidder 2 to be drawn from a more favorable distribution, requires a higher valuation than does bidder 2 to bid the same. For a second-price auction it suffices that each bidder submits his valuation. The seller's expected revenue in these two cases is shown in Table 4 for various values of the parameter α. Observe that the seller can realize an advantage from a first-price auction if there is substantial asymmetry.

Attrition Games

Closely related to auctions are contests to acquire an item in which the winner is the player expending the greatest resources. Riley (1988) describes several examples, including the 'war of attrition' that is an important model of competition in biological [Riley (1980), Nalebuff and Riley (1985)] and political [Wilson (1989)] as well as economic [Holt and Sherman (1982)] contexts. In attrition games, a player's expenditures accumulate over time as long as he is engaged in the contest; when all other players have dropped out, the remaining player wins the prize. Unlike an ordinary auction in which only the winner pays, a player incurs costs whether he wins or not; moreover, there need not be a seller to benefit from the expenditure. Some models of bargaining and arms races have this form, and price wars between firms competing for survival in a natural monopoly are similar.

Huang and Li (1990) establish a general theorem regarding the existence of equilibria for such games. We follow Milgrom and Weber (1985) to illustrate the construction of equilibria. In the simplest symmetric model, the n players' privately known valuations of the prize, measured in terms of the maximum stopping time that makes the contest worth the prize, are independent and identically distributed, each according to the distribution function F. A player with the valuation v obtains the payoff $v - t$ if the last of the other players drops out at time t and he does not, and otherwise it is $-t$ if he drops out at time t. Adopting a formulation in terms of distributional strategies (see §4), let $V(x) = F^{-1}(x) \equiv \sup\{v \mid F(v) < x\}$ and interpret $x = F(v)$ as the type of a player with the valuation v. Thus, a strategy σ_n that assigns a stopping time to each type implies a distribution $\rho_n = \sigma_n^{-1}$ of his stopping times. If each other player has this distribution of stopping times, then a player with the type x prefers to stop at time t

if his cost per unit time equals the corresponding conditional expectation of winning the prize. When k players remain this yields the condition:

$$1 = V(x)\frac{[k-1]\rho_k(t)^{k-2}\rho_k'(t)}{1-\rho_k(t)^{k-1}}\,, \tag{23}$$

since ρ_k^{k-1} is the distribution of the maximum of the other players' stopping times. The relevant case, however, is when only two players remain, which we now assume. Adding the equilibrium condition that $x = \rho_2(t)$ to the above condition yields a differential equation for the distribution ρ_2 that is subject to the boundary condition $\rho_2(0) = 0$. The solution in terms of the strategy is

$$\sigma_2(x) = \int_0^x \frac{V(z)}{1-z}\,dz\,. \tag{24}$$

Milgrom and Weber note further the properties that the hazard rate of the duration of the game is a decreasing function of time; the distribution of stopping times increases stochastically with the distribution of valuations; and the equilibrium is in pure strategies if and only if the distribution of valuations is atomless, in which case the stopping time as a function of the player's valuation is

$$\beta(v) = \int_{\underline{v}}^v \frac{t}{1-F(t)}\,dF(t)\,. \tag{25}$$

Extensions to asymmetric equilibria and to formulations with benefits or costs that vary nonlinearly with time are developed by Fudenberg and Tirole (1986) and Ghemawat and Nalebuff (1985).

4. Uniqueness and Existence of Equilibria

In this section we mention a few results regarding uniqueness of the symmetric equilibrium in the symmetric case, and regarding existence of equilibrium in general formulations. Affiliation is assumed unless mentioned.

Uniqueness

Addressing a significantly more general formulation (e.g., allowing risk aversion), but still requiring symmetry, Maskin and Riley (1986) derive a characterization similar to Theorem 1 of the symmetric equilibrium, and for the case of two bidders they establish that the symmetric equilibrium is the unique equilibrium. Thus, for symmetric first-price auctions there is some presumption that the symmetric equilibrium is the unique equilibrium. For symmetric second-price auctions, Matthews (1987) finds a symmetric equilibrium that with risk aversion generalizes Theorem 1 and for the common-value model this is shown to be unique by Levin and Harstad (1986). Milgrom (1981) shows, however, that symmetric second-price auctions can have many asymmetric equilibria, and Riley (1980) provides an example of a symmetric attrition game with a continuum of asymmetric equilibria. Bikhchandani and Riley (1991) provide sufficient conditions for uniqueness of the equilibrium in a second-price auction for the common-value model of preferences. For an 'irrevocable exit' version (an English auction with publicly observed irreversible exits of losing bidders) they establish that, within the class of equilibria with nondecreasing strategies, the unique symmetric equilibrium is accompanied (when there are more than two bidders) by a continuum of asymmetric equilibria with increasing and continuous strategies.

For auctions with asymmetrically distributed valuations, Plum (1989) proves existence as well as uniqueness within the entire class of measurable strategies for the case that there are two bidders, their valuations are independent and uniformly distributed (the support of bidder i's valuation is an interval $[0, \beta_i]$), and the sale price is a convex combination of the higher and lower bids. Smoothness of the equilibrium strategies is established for general independent distributions, and then for the case of uniform distributions and positive weight assigned to the higher bid, the equilibrium strategies are characterized by differential equations having a unique solution. The first-price auction maximizes the seller's expected revenue, which depends on the pricing rule if $\beta_1 \neq \beta_2$.

There appear to be no lower semi-continuity results indicating that, say, the symmetric equilibrium is the limit of equilibria of nearby asymmetric games. In particular, Bikhchandani (1990) studies a slightly asymmetric second-price auction in which there is a small chance that one bidder values the item more, but otherwise the auction is symmetric with common values; and for this game he finds that the equilibrium is strongly asymmetric. In a related vein, Bikhchandani and Riley (1991) provide sufficient conditions for the seller's revenue to be higher at the symmetric equilibrium of a second-price common-value auction than at any other.

Existence of Equilibria in Distributional Strategies

As usually formulated, games representing auctions pose special technical problems in establishing existence of equilibria. Primary among these is that such games are not finite, in that each bidder has an infinity of pure strategies. The source of this difficulty can be that infinitely many bids are feasible, or that an infinite variety of private information can condition the selection of a bid. It suffices in practice to suppose that only finitely many pure strategies are feasible, but this approach typically yields equilibria with mixed strategies, whereas often the corresponding game with a continuum of strategies has an equilibrium in pure strategies. A further characteristic feature of auctions is that payoffs are discontinuous in strategies. In §3, fn. 2, we mention an example of an auction with discriminating pricing that has no equilibrium, due essentially to the discontinuity of payoffs at tied bids. In simple formulations these features do not present difficulties, and as seen in §3, equilibria in pure strategies are characterized by differential equations.

Here we describe an alternative formulation that avoids some of these difficulties by generalizing the characterization in terms of differential equations for both auctions and attrition games. We follow Milgrom and Weber (1985) who introduced the formulation in terms of distributional strategies, but refer also to Balder (1988) who uses the standard formulation in terms of behavioral strategies.[8]

Standard formulations introduce pure strategies specifying actions at each information set, mixtures of pure strategies, and in extensive games, behavioral strategies that specify mixtures of actions at each information set. Static auctions have the special feature that for each play of the game each bidder takes an action, selection of a bid, at a single information set that represents his private information. Thus a pure strategy consists of a specification of a bid conditional on the observed private information. Alternatively in terms of Harsanyi's (1967) description of games with incomplete information, given his *type* as represented by his private information, each bidder selects a bid (or a stopping time). Given a distribution of his private information and a strategy (pure, mixed, or behavioral), therefore, each bidder's behavior is summarized by a joint distribution on the pair consisting of his information and his bid. Indeed, from the viewpoint of other bidders, this is the relevant summary. In general, we shall say that a *distributional strategy* is such a joint distribution for which the marginal distribution on the bidder's type is the one specified by the information structure of the game.

The formulation is specified precisely as follows. The game has a finite set N of players indexed by $i = 1, \ldots, n$, each of whom observes a type in a complete and separable metric space T_i and then takes an action in a compact metric space A_i of feasible

[8] An alternative approach, applied especially to attrition games, focuses on a detailed analysis of the role of discontinuous payoffs and establishes sufficient conditions for the limit of equilibria of a sequence of finite approximating games to be an equilibrium of the limit game; cf. DasGupta and Maskin (1986) and Simon (1987).

actions.[9] Allowing another complete and separable metric space T_0 for unobservable states, define $T = T_0 \times \ldots \times T_n$ and $A = A_1 \times \ldots, A_n$. The game then specifies each player's payoff function U_i as a real-valued bounded measurable function on $T \times A$, and the information structure as a probability measure η on the (Borel) subsets of T having a specified marginal distribution η_i on each T_i (including T_0). A distributional strategy for player i is then a probability measure, say μ_i, on the (Borel) subsets of $T_i \times A_i$ having the marginal distribution η_i on T_i. Each distributional strategy induces a behavioral strategy that is just a regular conditional distribution of actions given the player's type. Specified distributional strategies for all players imply an expected payoff for each player; consequently, a Nash equilibrium in distributional strategies is defined as usual. Milgrom and Weber impose the following regularity conditions:

R1 *Equicontinuous Payoffs:* For each player i and each $\epsilon > 0$ there exists a subset $E \subset T$ such that $\eta(E) > 1 - \epsilon$ and the family of functions $\{U_i(t, \cdot) \mid t \in E\}$ is equicontinuous.

R2 *Absolutely Continuous Information:* The measure η is absolutely continuous with respect to the measure $\hat{\eta} \equiv \eta_0 \times \ldots \times \eta_n$.

R1 implies that each player's payoffs are continuous in his actions, and therefore excludes known examples of auctions with finite type spaces that have no Nash equilibria in mixed strategies (cf. Milgrom (1979a) and Milgrom and Weber (1985)). However, it is sufficient for R1 that the action spaces are finite or that the payoff functions are uniformly continuous. R2 implies that η has a density with respect to $\hat{\eta}$. It is sufficient for R2 that the type spaces are finite or countable, or the players' types are independent, or that η is absolutely continuous with respect to some product measure on T.

Milgrom and Weber establish that with these assumptions there exists an equilibrium in distributional strategies, obtained as a fixed point of the best response mapping.[10] Moreover, with appropriate specifications of closeness for strategies, information structures, and payoffs, the graph of the equilibrium correspondence is closed (upper hemicontinuity).

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[9] Milgrom and Weber (1985, §6) define two natural metrics on the type spaces. Balder (1988) uses a formulation in terms of behavioral strategies and is able to dispense with topological restrictions on the type spaces.

[10] Balder (1988) extends this result by considering behavioral strategies. Without imposing topologies on the type spaces, and replacing R1 with the requirement that each player's payoff function conditional on each $t \in T$ is continuous on the space A of joint actions, Balder proves the existence of an equilibrium in behavioral strategies, and obtains an extension for a class of noncompact action spaces.

The relevance of these results for existence of equilibria in pure strategies is established in further results for the case that the marginal measure η_i of each player's private information is atomless.[11] First, if the action spaces are compact then for every $\epsilon > 0$ there exists an ϵ-equilibrium in pure strategies. For the second, follow Radner and Rosenthal (1982) in saying that a pure strategy σ_i purifies the distributional strategy μ_i if (a) for almost all of i's types the action selected by σ_i lies in the support of the behavioral strategy induced by μ_i (thus, the action is an optimal response); and (b) player i's expected payoffs are unchanged if i uses σ_i rather than μ_i's behavioral strategy. Part (b) is interpreted strictly: it must hold for every combination of the other players' distributional strategies.

THEOREM 4 (Milgrom and Weber): If R1 is satisfied and
(i) the players' types are conditionally independent given each state $t_0 \in T_0$, and T_0 is finite,
(ii) each player's payoff function is independent of the other players' types,
then each distributional strategy of each player has a purification. Moreover, the game has an equilibrium in pure strategies.

Except for the requirement that T_0 is finite, an application of this theorem is to the model discussed in §3, where the symmetric equilibrium in pure strategies was characterized exactly. Note that condition (ii) admits both the independent private-values model and the common-value model.

The intuitive motivation for Theorem 4 is simple. If a player's actions can depend on private information that is sufficiently fine (i.e., η_i is atomless), then from the viewpoint of other players his pure strategies are capable of generating all of the 'unpredictability' that mixed or distributional strategies might entail. Harsanyi (1973) follows a similar program in interpreting mixed strategies in complete information games as equivalent to pure-strategy equilibria in the corresponding incomplete-information game with privately known payoff perturbations.

5. Share Auctions

The theory of static auctions of several identical items provides direct generalizations of Theorem 1 that are reviewed by Milgrom (1981) and Weber (1983), some of which are

[11] Without invoking R1 or R2, this case already implies that each player's set of pure strategies is dense in his set of distributional strategies; cf. Milgrom and Weber (1985, Theorem 3).

summarized in §3, fn. 3, for the case that each bidder values only a single item. On the other hand, versions in which bidders' demands are variable pose rather different problems. Maskin and Riley (1987) show that for the seller an optimal procedure employs discriminating pricing of the form used in nonlinear pricing schemes. Here we address an alternative formulation studied by Wilson (1979) that preserves the auction format using nondiscriminating pricing.

If the supply offered by the seller is divisible then the rules of the auction can allow that each bidder submits a schedule indicating the quantity demanded at each price. For instance, if the seller's supply is 1 unit and each bidder i submits a (nonincreasing) demand schedule $D_i(p)$, then the clearing price p° is the (maximum) solution to the equation $D(p^\circ) = 1$, where $D(p) = \sum_i D_i(p)$.[12] Each bidder i receives the share $D_i(p^\circ)$ and pays $p^\circ D_i(p^\circ)$ if the pricing is nondiscriminating. [He pays an additional amount $\int_\infty^{p^\circ} p\, dD_i(p)$ if the pricing is discriminating, or in a Vickrey auction he receives a rebate $p^\circ - p_i - \int_{p_i}^{p^\circ} [D(p) - D_i(p)]\, dp$, where $D(p_i) - D_i(p_i) = 1$.]

To illustrate, consider a symmetric common-value model in which each bidder i observes privately an estimate x_i and then submits a schedule $D_i(p; x_i)$. Allowing risk aversion described by a concave utility function u, his payoff is $u([v - p^\circ]D_i(p^\circ; x_i))$ if the realized value is v. Assuming the bidders' estimates are conditionally independent and identically distributed given v, he can predict that if each other bidder uses a strategy D that is a decreasing function of the price, then the conditional distribution of the clearing price given the value v and his share y is

$$H(p; v, y) = \Pr\{ p^\circ \leq p \mid v, y \} = \Pr\{ \sum_{j \neq i} D(p; x_j) \leq 1 - y \mid v \} . \tag{26}$$

Consequently, a symmetric equilibrium requires that for each of his estimates x_i the choice of the function $y(p)$ that maximizes his expected payoff

$$\mathcal{E}\{ \int_{-\infty}^{\infty} u([v - p]y(p))\, dH(p; v, y(p)) \mid x_i \} \tag{27}$$

is $y(p) = D(p; x_i)$. The Euler condition for this maximization is

$$\mathcal{E}\{ u' \cdot [(v - p)H_p + yH_y] \mid x_i \} = 0 , \tag{28}$$

[12] This assumes no ask price a is imposed by the seller. Other allocation rules are possible; for example, the seller can choose the clearing price to maximize $[p - a]D(p)$ if pricing is nondiscriminating.

omitting the arguments of functions. Often, however, this condition allows a continuum of equilibrium strategies if the seller does not impose a minimum ask price.

An example is provided by omitting risk aversion ($u' = 1$) and assuming that (1) the marginal distribution of the common value v is Gamma with mean m/k and variance m/k^2, and (2) each observation x_i has the conditional distribution function e^{vx_i} on $(-\infty, 0)$. Then one equilibrium is

$$D(p;x) = \frac{1}{n-1}[1 - 2p\frac{k - nx}{n(n+m)}], \qquad (29)$$
$$p^\circ = \tfrac{1}{2}\frac{m+n}{k - \sum_i x_i} = \tfrac{1}{2}\mathcal{E}\{v \mid x_1, \ldots, x_n\}.$$

The clearing price is positive, as is each bidder's resulting share. Note that in this example the clearing price is half the conditional expectation of the common value, regardless of the number of bidders. Anomalies appear in many examples of share auctions; presumably better modeling of the seller's behavior is necessary to eliminate these peculiarities.

One motive for studying share auctions is to develop realistic formulations of 'rational expectations' features in markets affected by agents' private information. The Walrasian assumption of price-taking behavior can be paradoxical in such markets: if demands reflect private information then prices can be fully informative, but if agents take account of the information in prices then their demands at each price are uninformative. By taking account of agents' effects on the clearing price, models of share auctions avoid this conundrum. Jackson (1988) develops this argument and shows further the incentives that agents have to obtain costly information.

A share auction in the case of a finite number of identical items offered for sale is just a multi-item auction with nondiscriminating pricing: bidders whose offers are accepted pay the amount of the highest rejected bid. This formulation is developed by Milgrom (1981), who uses a symmetric model and the symmetric equilibrium identified in Theorem 1 to establish the information-revealing properties of the transaction price. He shows that bidders nevertheless utilize their private information in selecting a bid, and have incentives initially to acquire information. Thus, this formulation provides a sensible alternative to the price-taking behavior assumed in Walrasian models of rational-expectations equilibria.

6. Double Auctions

In a static double auction, both the sellers and the buyers submit supply and demand schedules. A clearing price is then selected that equates supply and demand at that price. If the pricing is nondiscriminating then all trades are consummated at the selected clearing price. This procedure is sometimes called a demand-submission game.

Working with a complete information model of Walrasian general equilibrium, Roberts and Postlewaite (1976) anticipate the subsequent game-theoretic analyses of double auctions. They consider a sequence of finite exchange economies (each described by a simple measure μ_n on the set of agents' characteristics) converging to an infinite economy (required to be a measure) at which the Walrasian price correspondence is continuous. They establish the following property for each agent persisting in the sequence whose inverse utility function is continuous in a neighborhood of the Walrasian prices for the limit economy: For each $\epsilon > 0$ there exists N such that if $n > N$ then the agent can not gain more than ϵ from submitting demands other than his Walrasian demands. The gist of this result is that in a large economy an agent's incentive to distort his demand to affect the clearing prices is small. Subsequent work has examined whether a comparable result might hold for Nash equilibria, especially if agents' characteristics are privately known; however, comparable generality in the formulation has not been attempted.

More detailed characterizations of static double auctions have been obtained only for the case that a single commodity is traded for money, each seller offers one indivisible unit and each buyer demands one unit, their valuations are independent and (among sellers and buyers separately) identically distributed on the same interval, the traders are risk neutral, and the numbers of sellers and buyers are common knowledge. If the clearing price is p then a trader's payoff is $p - v$ or $v - p$ for a seller or buyer with the valuation v who trades, and zero otherwise. If the asks and bids submitted allow k units to be traded, then the maximum feasible clearing price is the minimum of the k-th highest bid and the $k+1$-th lowest ask, and symmetrically for the minimal clearing price. A symmetric equilibrium comprises a strategy σ for each seller and a strategy ρ for each buyer, where each strategy specifies an offered ask or bid price depending on the trader's privately known valuation. Following Myerson (1981), to avoid mixed strategies it is useful to assume the 'regular' case that the distribution, say F, of a trader's valuation has a positive density f and that $v + F(v)/f(v)$ is increasing for a seller or $v - [1 - F(v)]/f(v)$ is increasing for a buyer. The symmetric equilibrium pure strategies are characterized by differential equations in Satterthwaite and Williams (1989abc), Williams (1987), and Wilson (1985). The most general characterization and proof of existence, in terms of vector fields for generic data and symmetric equilibria, is by Williams (1988).

The basic characterization of the effect of many traders is due to Satterthwaite and Williams (1989abc) and Williams (1988), who address the case that the clearing price used is the maximal (or symmetrically, the minimal) one. In this case the sellers' dominant strategy is the identity $\sigma(v) = v$, because a seller's ask can not affect the price at which she trades. Their main result demonstrates for each buyer's valuation v that $v - \rho(v) = O(1/M)$, where M is the minimum of the numbers of sellers and buyers. Thus, in double auctions of this kind with many traders of both types, each trader asks or bids nearly his valuation; and, the resulting allocation is nearly efficient, since missed gains from trade are both small and unlikely.[13] For example, if all valuations are uniformly distributed on the unit interval and there are m buyers and n sellers, then the unique smooth symmetric equilibrium is linear, $\rho(v) = \frac{m}{m+1}v$, independently of the number of sellers.

Williams (1991) shows further that if there is a single buyer then his strategy is independent of the number of sellers, whereas if there is a single seller and a regularity condition is imposed then the buyers' strategy again implies bids that differ from their valuations by $O(1/m)$. These results indicate that asymmetry in the auction rule leads to competition among the buyers that is the main explanation for the tendency towards *ex post* efficiency as the number of buyers increases.

Chatterjee and Samuelson (1983) construct a symmetric equilibrium for the case of one seller and one buyer with uniformly distributed valuations, taking the transaction price to be the midpoint of the interval of clearing prices: $\sigma(v) = \frac{1}{4} + \frac{2}{3}v$ and $\rho(v) = \frac{1}{12} + \frac{2}{3}v$. However, Leininger, Linhart, and Radner (1989) show that this linear equilibrium is one among many nonlinear ones; indeed, the characterization by Satterthwaite and Williams (1989abc) shows that this feature is entirely general. Myerson and Satterthwaite (1983) establish, nevertheless, that with this linear equilibrium the double auction is *ex ante* efficient; that is, no other (individually rational — each trader's conditional expected payoff given his valuation is nonnegative) trading mechanism has an equilibrium yielding a greater sum of the two traders' expected payoffs. This conclusion does not extend to greater numbers of sellers and buyers, however; cf. Gresik (1991a).

Wilson (1985ab) examines the weaker criterion of *interim* efficiency defined by Holmström and Myerson (1983); namely there is no other trading mechanism having an equi-

[13] This shows also that this property must hold for any optimal trading mechanism, which strengthens a result in Gresik and Satterthwaite (1989).

librium for which, conditional on each trader's valuation, it is common knowledge that every trader's expected payoff is greater. Using the model of Satterthwaite and Williams, except that the supports need not agree and the clearing price can be an arbitrary convex combination of the endpoints, and assuming that the derivatives of the strategies are uniformly bounded, he demonstrates that a double auction is *interim* efficient if M is sufficiently large.

Significantly stronger results are obtained by McAfee (1989) for double auction rules that allow a surplus of money to accumulate. In the simplest of the three versions he examines, the rules are as follows. Suppose the bids and offers submitted allow at most a quantity q to be traded; i.e., q is the maximum k such that the k-th highest bid b_k exceeds the k-th lowest offer s_k. Then $q-1$ units are traded with the $q-1$ highest bidders buying items at the price b_q, and the $q-1$ lowest offerers selling items at the price s_q. Note that a monetary surplus of $(q-1)(b_q-s_q)$ remains. If these rules are used then the traders have dominant strategies, namely, bid or offer one's valuation. Only the least valuable efficient trade is lost. In fact, for a slightly more complicated scheme, with n traders the realized prices differ from an efficient price by $O(1/n)$ and the loss in expected potential surplus is approximately $O(1/n^2)$ — in the sense that it is $O(1/n^\alpha)$ for all $\alpha < 2$.

Bid-Ask Markets

Dynamic double auctions in which buyers and sellers have repeated opportunities to submit or accept bids and offers are commonly used in commodity markets and some financial markets. They have been intensively studied experimentally as we describe in §8, but few theoretical analyses have been published. Their remarkable efficiency in realizing gains from trade even when subjects have little information has motivated two studies arguing that simple heuristic behaviors [Easley and Ledyard (1982)] or arbitrage processes [Friedman (1984)] could explain the data.

Friedman's (1984) analysis supposes that the traders' strategies imply 'no congestion' at the conclusion of trading; that is, if an extra static double auction were appended to the end of the bid-ask market, and for this auction each trader were to assign positive probability that the final maximum bid and minimal ask would be acceptable to some other traders, then no trader would actually want to accept one of these offered trades nor would any trader want to alter his bid or ask — thus, no trade would occur in the appended auction. Using this auxiliary assumption instead of the usual requirement of

Nash equilibrium, he shows that if the commodity traded is divisible (and preferences are regular) then the final bid and ask must agree and be a market clearing price; in particular, the attained allocation is efficient. If the items traded are indivisible then this conclusion is slightly weakened: the allocation is within a single trade of being efficient. Although it is not explained precisely how traders' strategies achieve the no-congestion property, Friedman's results demonstrate that fairly weak properties of traders' strategies suffice to explain the remarkable efficiency observed experimentally, and that perhaps it is not necessary to appeal to complete analyses of Nash equilibria.

McAfee (1989) offers a different view based on a dynamic auction design derived from his modified double auction rules described previously (a similar design, but without monetary surpluses, is studied by McCabe *et al.* (1989, 1990a)). Because this auction is plausibly similar to a bid-ask market, and its unrealized gains from trade are approximately of order $O(1/n^2)$, the experimental efficiency of bid-ask markets is perhaps unsurprising.

Wilson (1986) proposes a conjectured equilibrium and verifies that it satisfies various necessary conditions (as well as the no-congestion property). His 'equilibrium' is a multilateral generalization of the equilibrium for bilateral bargaining constructed by Cramton (1984, 1990) in which at each time the buyer with the highest valuation and the seller with the lowest valuation are endogenously matched into a bargaining process that is affected by subsequent opportunities to trade and by the prospect that immediately profitable trades might be usurped by competing traders. In particular, the risk of usurpation plays the role of the interest rate commonly used in studies of bargaining to reflect impatience to trade early. During the bargaining process, each party delays making a serious bid or offer (i.e., one with positive chances of acceptance) sufficiently to signal credibly his valuation, and after a serious offer the other party also delays sufficiently to signal credibly before accepting or making an offer that is surely accepted. However, this model fares poorly in explaining the data that subjects often do not trade in order of their valuations, and often trades are completed by extra-marginal traders (e.g., buyers with valuations less than the clearing price). A variety of other bargaining models and associated equilibria are available [Kennan and Wilson (1989)], nevertheless, and perhaps one of these could fit the data better.

Markets conducted by intermediaries such as specialists have also been studied. One strand of research focuses on how a specialist can cope with traders having superior ('inside') information. Glosten and Milgrom (1985) characterize the specialist's bid and ask prices that account for the effects of adverse selection. For a double auction in which the specialist trades for his own account to set the clearing price, Kyle (1985) derives the equilibrium between the specialist's pricing strategy and the insider's strategy of modulating his trades to avoid revealing too much information too early. The finance literature includes many subsequent studies.

7. Applications

Auctions in which a single seller offers one or several items for sale are common; Cassady (1967) describes many examples. He also notes that different types of auctions tend to be associated with particular kinds of commodities. For example, oral auctions, either English (ascending bids) or Dutch (descending offers), are favored for animal stock and perishable commodities, perhaps to ensure rapid consideration of many lots with variable quality attributes. Most auctions of art and antiques use the oral format, as do sales of property, used machinery, and other producers' durables. On the other hand, in the U.S. new issues of corporate bonds and stock are usually sold to investment bankers via sealed bids, as are rights for timber and minerals, including coal and oil. Land and buildings are often sold via sealed-bids also. In many countries, large firms and government agencies use sealed-bid auctions to select vendors and to procure services, especially construction. Rozek (1989) describes the increasing use of auctions to select providers of electric power supplies. In practice most auctions use discriminating pricing rules, but exceptions include the Exxon Corporation's auctions of its bonds (Levinson (1987)) and the auctions of privately placed preferred stock conducted by Goldman Sachs & Company (1987) and others. Brown (1984, 1986, 1987, 1989), Engelbrecht-Wiggans (1987b), Holt (1979, 1980), Laffont and Tirole (1987), Lang and Rosenthal (1990), McAfee and McMillan (1985, 1987c), and Nti (1987), as well as the seven contributed chapters in Part V of Engelbrecht-Wiggans, Shubik, and Stark (1983), are indicative of studies of procurement contracting that take account of incentive effects in contract design as well as strategic behavior in the auction process. Kahn *et al.* (1990) report on a simulation study of auctions for procurement of power supplies in the electricity industry. The design of optimal auction rules for efficient procurement contracting is characterized by Riordan and Sappington (1987).

The sealed-bid auctions studied most thoroughly are those conducted periodically since 1954 in the U.S. to sell exploration and development leases for tracts on public lands and offshore on the outer continental shelf (OCS). These leases are unusual for their value (often sold for tens, and occasionally hundreds, of millions of dollars) and the auctions are notable for the evident intensity of strategic behavior; indeed, the larger oil companies maintain large permanent staffs for the preparation of bids. The leases have an evident common-value component because the amount and value of oil and gas is essentially the same for all bidders [Capen, Clapp, and Campbell (1971)]; consequently, firms have increasingly realized that bidding strategies must be carefully designed to avoid the effects of adverse selection. That, each bidder must take account of the so-called 'winner's curse' that the one who most overestimates the value of a lease is the one most likely to win. Experimental and empirical evidence on the incidence of overbidding in common-value auctions is summarized in §8. Strategic models of bidding have long been used by the Department of Interior to examine public policies regarding leasing [DeBrock and Smith (1983), Reece (1978, 1979), Wilson (1981)] and by the Department of Defense regarding procurement [Engelbrecht-Wiggans, Shubik, and Stark (1983, Part V)]. Accounts of the oil companies' use of strategic analysis are unusual because of the extreme secrecy they maintain.

Share auctions are rare in practice. In most countries, public issues of firms' stocks are sold as single blocks (often via auctions) to investment banking firms who then resell the stock to investors, perhaps because marketing is an important ingredient. This has also been the practice for bonds [Christenson (1961)], which for public utilities are invariably sold via sealed-bid first-price auctions. A recurring share auction is conducted by the Paris Bourse for stocks of firms newly listed on the exchange; however, it rations shares to bidders and rejects very high bids, which apparently is necessary because the transaction price is chosen to be less than the clearing price to attract bidders [Jacquillat and McDonald (1974)]. In the 1970s the International Monetary Fund sold its excess gold supplies via share auctions; and in this period there were recurrent proposals to sell shares of large 'unitized' offshore oil leases. In the 1980s, shares of several privatized national corporations were sold via procedures resembling share auctions.

The U.S. Treasury conducts weekly sales of bonds via a multi-unit auction that closely resembles a share auction. Cammack (1991) describes the institutional aspects of this market and the secondary resale market; she also provides evidence that the market is affected by dispersed private information among the bidders. Except for a few trials, the Treasury Bill auction has used discriminating pricing. Plott (1982) reports experimental evidence that discriminating and nondiscriminating pricing in share auctions yield about the same revenue to the seller. Bikhchandani and Huang (1989) provide a novel analysis of the Treasury auction, for both the discriminating and nondiscriminating pricing rule. Two important institutional features are that (1) the Treasury accepts 'noncompetitive bids' filled at the average price of the accepted competitive bids, and (2) the competitive bidders (who are dealers purchasing in order to resell in a secondary market) have incentives to *signal* their private information in order to influence the subsequent price in the secondary market. Whereas the first feature adds a noise component to the primary auction and the secondary market, the second feature increases the symmetric bidding strategy by an extra term for which Bikhchandani and Huang obtain a closed-form expression. For discriminating pricing, a stronger property than affiliation, interpreted as complementarity of information, is required to establish existence. Moreover, revelation of information by the seller need not increase expected revenues. They also establish sufficient conditions for the symmetric equilibrium of the nondiscriminating auction to be preferred by the seller.

Static double auctions, relying on sealed bids and offers, are employed frequently in security markets. They are used to determine the opening price in stock exchanges and markets for precious metals. They are also used in periodic markets for trading privately placed preferred stock [Goldman Sachs & Co. (1987)]. Proposals to automate trading on the major exchanges for financial instruments have focused on using periodic (e.g., hourly) double auctions to accomplish market clearing. The Stockholm Exchange's automated procedures resemble an English auction, however.

In terms of trading volume, however, dynamic versions of double auctions are more widely used. In most organized exchanges, markets for storable commodities, industrial metals, and crude oil are conducted via open outcry of bids and asks from floor traders and brokers acting for clients. Summaries of experimental results on the extraordinary efficiency of these markets are reported by Plott (1982) and Smith (1982). Financial markets mostly rely on intermediaries such as specialists who maintain inventories and order books of bids and offers in order to sustain continual trading opportunities and price stability, but some markets for options and futures contracts use oral bid-ask auctions; cf. Brady (1988) and Miller (1987) for details of the organization of security markets in the U.S.

Labor markets involve interesting variants of double auctions to match workers with positions in firms, students with openings at schools, etc. Called the 'marriage problem', this version differs in that each buyer or seller offers an item with unique quality attributes valued differently by each party on the other side of the market. Much of the literature focuses on a procedure that provides one side of the market a dominant strategy. One dynamic procedure has buyers apply to sellers, who reject or tentatively accept each application, and this continues until each buyer is accepted or would prefer not to trade with any remaining seller. Excluding strategic behavior by the sellers (i.e., they accept their preferred applicants), this procedure provides a dominant strategy for the buyers, and yields for them their best allocation in the core of the associated cooperative game. Roth (1984abc) and Roth and Sotomayor (1990) describe the history of the market in the U.S. that matches medical interns (sellers) with positions in hospitals (buyers), which has developed a similar procedure to cope with hospitals' incentives to act strategically. For a version in which prices as well as the matching are determined by the process see Demange and Gale (1983). Game-theoretic analyses of Nash equilibria have not been fully developed.

8. Experimental and Empirical Evidence

Game-theoretic models of auctions make strong assumptions about the information and behavior of participants. Correspondingly, the predictions obtained from the models imply severe restrictions on the outcomes of auctions in practice and in experimental settings. Students of economic behavior relying on empirical data or experimental observations have therefore found auctions to be rich sources of evidence. In this section we review briefly several principal studies. The discussion divides between experimental studies in which the objective is to examine behavior conditional on controlled environments, and empirical studies in which much of the relevant data about the environment is inaccessible.

Experimental Studies

The objective of experimental studies is to examine subjects' bidding behavior in settings in which the experimentor has complete information about the economic data (but not subjective aspects of subjects' preferences, such as risk aversion), and the procedural rules and informational conditions are controlled. Results from experiments allow comparisons of procedural rules and other features, and most importantly, they allow tests of the hypothesis that *all* participants simultaneously use equilibrium strategies. A large portion of the experimental evidence is summarized in three surveys by Smith (1982, 1987) and Plott (1982) that we distill even further. Roth's (1988) survey is an incisive methodological critique.

Smith's (1987) summary of over 1500 single-item auction experiments in which subjects have independent private valuations concludes that English (oral discriminating ascending) and second-price (static nondiscriminating) auctions achieve high efficiency and have about the same mean observed prices, whereas first-price (static discriminating) and Dutch (oral discriminating descending) auctions have appreciably lower efficiency and appreciably higher prices (first-price auctions have higher efficiency measures and higher prices than Dutch auctions). These differences are attributed to risk aversion, and with this proviso, judged to be consistent with Nash equilibrium; further qualified support is found in experiments with multi-item discriminating auctions (but not for nondiscriminating versions).[14] Smith's (1982) summary emphasizes further that heterogeneity of risk aversion among subjects is necessary to explain the data, which exhibit considerable dispersion; a critique and alternative view is suggested by Harrison (1989). Plott (1982, p. 1505) describes experiments with multi-unit static discriminating auctions and concludes that the results "provide support for Nash equilibrium bidding models when there are several (three or four) bidders", and that "after convergence [of subjects' behaviors] takes place, [discriminating and nondiscriminating auctions] generate about the same revenue" as implied by theoretical results.

The extensive literature on double auctions, especially bid-ask markets (oral discriminating multi-item), is partially reviewed by Smith (1982, §3B) and Plott (1982, §II); for more recent results see Friedman and Ostroy (1989). Although portions of this work involve a variety of particular institutional features, the main finding emphasized in all studies is that transaction prices usually converge rapidly (say, 4 repetitions), even with few participants (8), to Walrasian clearing prices.[15] In addition, the efficiency of trading is very high.[16] This striking finding is quite robust, but its conformity to the predictions of game theory is mute due to the dearth of theoretical results.

Single-item static discriminating auctions with common values have been studied experimentally by Kagel and Levin (1986). Their main conclusions are that "in auctions involving a limited number of bidders (3 – 4 bidders), average profits are consistently

[14] Kagel and Levin (1988) study the role of risk aversion in experiments using a different experimental design based on a 'third-price' auction that provides a stronger test.

[15] This is also true when rational expectations aspects are involved, provided a sufficiently rich set of securities are traded, as shown by Plott and Sunder (1988).

[16] Asymmetric versions, say with a single seller, often do not achieve monopoly outcomes as might be predicted, but rather approximate the more nearly Walrasian outcome predicted by the Coase property; cf. Gul, Sonnenschein, and Wilson (1986).

positive and closer to the Nash equilibrium bidding outcome than to the winner's curse hypothesis;" in particular, profits average about two-thirds of the amount predicted by the equilibrium strategies for risk-neutral bidders. However, "bids are found to be an increasing function of the number of rivals faced, in clear violation of risk-neutral Nash equilibrium bidding theory", contributing to a "reëmergence of the winner's curse, with bankruptcies and negative profits, in auctions with large numbers (6 – 7) of bidders." [17] They further observe that providing public information about the common value increased the seller's revenue in the former case (few bidders) as predicted, but decreased it in the latter. All of these conclusions refer to auctions with experienced bidders, and in particular they emphasize that learning is evident and partially successful in repeated auctions with few bidders, but not in auctions with many bidders; moreover, the learning is specific, in that it is not entirely carried over to new situations. They conclude, therefore, that the probable explanation of the results is persistent errors in judgment, manifested in an inability to fully comprehend the adverse selection that afflicts bidders in common value auctions.

Empirical Studies

Empirical studies must contend with less complete data and few controls on the auction environment are possible. On the other hand, they have the advantage that the data pertain to practical situations in which the stakes are often large and the participants are skilled and experienced. In the case of auctions of offshore oil leases, millions of dollars are at stake and the firms bidding for leases have large staffs, data bases, and computer facilities devoted to the task of preparing bids. The procedural rules are simple, since the lease is awarded to the high bidder at the price offered by sealed tender, but the information structure is not, and in important cases is very asymmetric. In view of the extensive review of empirical studies in McAfee and McMillan (1987), we review only studies omitted from their survey.

We mentioned above the experimental finding that even experienced bidders tend to overbid in common-value auctions with six or more bidders. Hendricks, Porter, and Boudreau (1987) examine a similar situation in auctions of leases for wildcat tracts (i.e., in unexplored areas) for the years 1954 to 1969. One of their main conclusions is that winning bidders' average realized net profits were negative for auctions with more than

[17] These results are replicated for subjects who were professional managers of construction firms in Dyer, Kagel, and Levin (1989a); and for second-price auctions, by Kagel, Levin, and Harstad (1988).

six bidders. The authors point out that the data can be explained by non-optimal bidding strategies that account inadequately for adverse selection in valuation estimation, or equally, by adverse selection in estimating the number of bidders. That is, most tracts receive fewer than six bids (the average was 3.5 in the sample) and supposing that firms expect this, profits will be less on those tracts receiving more bids. Overall, winning bidders captured about a quarter of the value of the tracts, which is consistent with a supposition that active bidders expected three or four bids to be submitted. Overall, the authors conclude that "the data are consistent with both the assumptions and predictions of the [common value] model," allowing for bidders' uncertainty about the number of active bidders in each auction.[18]

These data are from the period before the publication of Capen, Clapp, and Campbell's (1971) influential article suggesting that adverse selection (the winner's curse) might account for low returns realized by winning bidders in such auctions. Helfat (1987), using *ex ante* data on expected returns and allowing risk aversion in a portfolio model of firm's decisions, finds that returns remained low until the OPEC oil embargo of 1974, but rose substantially thereafter due to lower average bids. Whether this effect is associated with better bidding strategies or is an incidental consequence of the altered structure of the oil market is unclear.

Auctions of 'drainage' tracts adjacent to explored tracts typically involve substantial asymmetries of information, since firms who have explored neighboring tracts have superior information. Hendricks and Porter (1988) derive seven implications of the equilibrium bidding strategies for the case that one bidder has superior information, as in the common-value model presented in §3. They test these predictions using data on 114 drainage tracts leased in the period 1959 to 1969. Their conclusion, subject to one proviso, is that essentially all seven of these predictions are confirmed by the data. The proviso is that multiple neighbors act as a single cartel to submit a single serious bid; indeed, at the time cartels were not prohibited and firms routinely cooperated in other activities, and Hendricks and Porter provide substantial evidence of coordinated bidding by neighboring firms. The sharpest test is the prediction that non-neighbor firms' profits should be zero on average, compounded from positive profits when a neighbor bid and lost and negative profits when neighbors chose not to bid: this prediction is confirmed, and in particular average profits differed from zero by only one-quarter standard de-

[18] This conclusion is stronger than in previous studies, where mixed results were often reported; for example, see Gilley, Karels, and Leone (1986) and the references therein.

viation of the sample mean. Overall, the authors "find that the data strongly support the hypotheses that ... firms bid strategically in accordance with the Bayesian-Nash equilibrium model." Hendricks, Porter, and Spady (1988) obtain similar results for data from the period 1970 — 1979, using however a somewhat richer model that assumes (realistically) that the government uses a random reservation price. The conclusion is again that "the hypothesis that neighbor and non-neighbor firms bid strategically in accordance with the theory of auctions with asymmetric information is strongly supported by the data." This analysis is extended further by Hendricks, Porter, and Wilson (1992) to take account of the informational content of the government's reservation price: assuming affiliation, they show that the distribution of the informed (neighbor) bidder's bid stochastically dominates the distribution of the highest bid among those submitted by uninformed bidders, but conditional on the bid being high enough these two distributions are identical. The data from drainage tracts favor rejection of a null hypothesis that the two marginal bid distributions are identical, and acceptance of the null hypothesis that the two conditional distributions are identical. These positive conclusions are reversed for less risky 'development' tracts, providing further support for the supposition that informational differences account for the results.

Thiel (1988) uses data from 130 auctions of highway construction contracts by 28 state governments in the U.S. to test the common-value model with symmetric information, assuming Normal probability distributions. He supposes that the state engineer's estimate of the cost of fulfilling the contract, revealed after the auction, is an unbiased estimator of the true (but unobserved) cost. However, his main conclusion that "the model fits the data reasonably well" is suspect since the equilibrium bidding strategies are misspecified, as shown by Levin and Smith (1991).

9. Comparisons of Auction Rules

Several studies have compared the distributional effects of various types of auctions. Here we provide a sample of results that indicate major themes.

In §3 we mentioned for single-item auctions in which bidders are risk-neutral and have independently and identically distributed private valuations, that the seller and the bidders are indifferent among the standard procedural rules, such as first-price or second-price and oral or sealed bids. Matthews (1987) shows that this feature persists for bidders in the case that all bidders have identical exponential utility functions, so that each has the same constant Arrow-Pratt measure of risk aversion, even if the number of bidders is uncertain — although the expected price is higher in a first-price auction if the number is not revealed, as shown by McAfee and McMillan (1987b) and studied experimentally by Dyer, Kagel, and Levin (1989b). However, if the risk-aversion measure is decreasing then the bidders prefer (SP) a second-price auction to (FPR) a first-price auction with the number of bidders revealed to (FPU) a first price auction with the number not revealed, in that order; moreover, the seller's preference is the reverse ordering if the seller is risk-neutral.[19] However, if their valuations are affiliated, then the bidders' preferences are biased away from the second-price auction; e.g., if utilities are exponential then they prefer FPR to FPU to SP, whereas the seller prefers FPU to FPR to SP.

Milgrom and Weber (1982), assuming that the number of bidders is known and that bidders are symmetric and their valuations are affiliated, establish that an 'English' oral ascending auction has a higher expected price than a second-price auction, and in turn if bidders are risk-neutral, the latter is higher than the expected price in a first-price auction. In all three auctions, moreover, a risk-neutral seller prefers to (acquire and) disclose any private information it can if doing so is costless.[20] The comparisons are applications of the 'linkage principle': since bidders' profits are returns to their private information, procedures that reveal more of their private information during the auction (such as an English auction), or that dilute their information (such as revelation of information known to the seller), tend to increase the expected price obtained. Milgrom and Weber also show for each of these auctions that between two such auctions that differ only in the ask price and an entry fee but attract the same set of bidders, the one with the lower ask price and higher entry fee obtains the higher expected price for the seller.

Assuming two bidders, Hausch (1987) shows that these results extend to common-

[19] These results assume that the number of active bidders is uninformative about their valuations and the seller's ask price is not binding. Holt (1980), Harris and Raviv (1981b), Maskin and Riley (1984), and Riley (1989) show that the seller prefers a first-price sealed-bid auction to an English auction if bidders are risk averse, a preference that is strengthened if the seller is also risk averse. Maskin and Riley also investigate a plethora of other schemes the seller can use to exploit the bidders' aversion to risk.

[20] These results are valid also for an ask price that is the same in all three auctions, and the seller's preference for an English auction over a second-price auction carries over to the case that bidders have exponential utilities. Riley (1985) shows further for the comparison between first and second-price auctions, that of two rules using a price that is a weighted combination of the first and second bids the one giving greater weight to the second has the higher expected price; and indeed, the seller prefers rules that weight all bids as versus only the high bid. However, Maskin and Riley (1984) show for the case of independent private valuations that wealth effects can make an English auction an inferior choice for the seller. Riley (1989) uses elementary methods to examine some of these features.

value auctions in which the bidders' sampling distributions differ but their conditional distributions (of one bidder's sample given the other's observation) agree; moreover, an equilibrium with symmetric strategies exists. However, if the conditional distributions disagree then the strategies may be asymmetric and the first-price auction can have a higher expected sale price.

For common-value auctions, Harstad (1990) extends Milgrom and Weber's results to contexts in which the number of bidders is determined endogenously by the bidders' participation cost (e.g., the cost of acquiring sample information) and the seller's selection of the auction type (including the information revealed, the entry fee, and the ask price), provided the induced probability of selling the item is sufficiently large. Again with this proviso, an auction format for which participation costs are recovered with fewer participants is preferable for the seller. Similarly, a reduction in participation cost is preferable if the number of participants responds inelastically. Both of these observations follow from the general principle that with endogenous participation, and conditional on a sale, the seller's expected revenue is the expected common value less the aggregate of participation costs. Thus, in this context the role of the linkage principle is played by the induced reduction in participation.

Applications to bidding for oil leases have also noted that royalties payable on the actual amount extracted can alleviate bidders' risk aversion and therefore increase the seller's expected revenue. Riley (1985) shows further, subject to regularity conditions, that even without risk aversion, if bidders' valuations are affiliated then contingent payment schemes conditioned on *ex post* observations are optimal for the seller. Nevertheless, high royalties can diminish the winning bidder's incentive to pursue an efficient plan of oil recovery, and therefore the net effects are mixed.

The seller's ask price is affected by the power to commit in advance. To illustrate, consider only the case that the bidders have independent private valuations. In a static auction with a fixed number of bidders, the seller usually prefers to commit to an ask price above cost. But in the Dutch auction dynamic version, having failed to receive a bid the seller prefers to continue lowering the price so long as it remains above cost, which increases the efficiency of trade. Engelbrecht-Wiggans (1988) studies an example modeled after an historical incident: assume that the number of bidders increases until the expected gain *ex ante* from attending an auction is reduced to the expense incurred. In this case, the seller prefers *ex ante* to commit to an ask price that is lower than the expectation of the ask prices conditional on the number of attendees, chosen to take account of the option of reoffering the item in a fresh auction with a new sample of bidders. The reason is essentially that a low ask price *ex ante* attracts more bidders and raises the average sale price. Engelbrecht-Wiggans cites an incident in which taxation imposed on goods offered for sale, rather than those actually sold, allegedly increased tax revenues as well as benefited sellers — the reason being that taxation of offered goods provides a disincentive to reoffer an item, and therefore lowers conditional ask prices to levels closer to the *ex ante* optimum.

Auctions in which the seller has superior information about an item of common value to the bidders have novel features. Vincent (1990) studies a repeated auction in which only the seller knows the benefit each of two identical uninformed bidders would obtain from acquiring the item, and the seller's valuation is a fixed amount less than the bidders' valuation; also, all parties discount delayed payoffs. The sequential equilibrium in this case involves 'screening' by the bidders: they offer an increasing sequence of bids until the seller accepts. The bidders' expected profits are zero of course, and the seller captures the difference, but the outcome is inefficient because of the delay. In Vincent's example, the seller prefers to exclude repetition of the auction, or to deal with only one bidder rather than two; in both cases the seller's motive is to reduce screening and the resulting delay.

10. Optimal Auctions

In parallel with the analyses of specific auction forms, a large literature has addressed the design of auction rules that are efficient or optimal for the seller. In this section we provide a brief synopsis of the main results. If not mentioned, the default assumption is that bidders have independently and identically distributed valuations, according to a distribution function having an increasing hazard rate.

The basic results are due to Myerson (1981) and Myerson and Satterthwaite (1983), although here we follow the exposition in Wilson (1985ab).[21] An alternative approach

[21] Our exposition excludes the most general case in Myerson by assuming, in effect, a monotone hazard rate. Other authors developing this methodology are Harris and Raviv (1981a), who derive an optimal priority pricing scheme similar to a Dutch auction; Harris and Raviv (1981b), who obtain specializations of Myerson's results; Harris and Townsend (1981), who study the general properties of revelation mechanisms; and Moore (1984). Extensions to the case that a seller offers a quantity of a (divisible) good and each buyer can select any amount to purchase are developed by Maskin and Riley (1986); in this case the seller's optimal mechanism entails a nonlinear price schedule. Applications to the design of auctions of procurement or franchise contracts are developed by Riordan and Sappington (1987).

by Bulow and Roberts (1989) shows that the methodology is isomorphic to the theory of a monopolist offering prices that discriminate among multiple markets.

As in the method of distributional strategies in §4, assume that each participant i 's valuation (for trade of a single indivisible item) depends on his privately known type t_i via a decreasing function $u_i(t_i)$ if i is a buyer and an increasing function $v_i(t_i)$ if i is a seller, where the types are independently and uniformly distributed on the unit interval. Assume further that $\bar{u}_i(t) = tu_i(t)$ and $\bar{v}_i(t) = tv_i(t)$ are concave and convex functions respectively, as implied by the increasing hazard rate property. Party i 's expected payoff from an equilibrium of a particular mechanism is denoted $V_i(t_i)$, and we consider a welfare measure

$$\mathcal{W} = \mathcal{E}\left\{ \sum_i \alpha_i(t_i)V_i(t_i)\right\}, \tag{30}$$

depending on nonnegative welfare weights $\alpha_i(t_i)$ that may depend on the party's type, as in the case of *interim* incentive efficiency (Holmström and Myerson (1983)). The 'revelation principle' takes advantage of the property of an equilibrium that each party must prefer to act according to his true type to conclude, say for a buyer, that

$$V_i(t_i) = V_i(1) + \int_{t_i}^1 P_i(\hat{t})\, d[u_i(0) - u_i(\hat{t})], \tag{31}$$

where $P_i(\hat{t})$ is the probability that i trades if his type is $\hat{t}$. Using this property, along with the feasibility condition that net trades of money and goods among the parties must sum to zero, enables one to rewrite the welfare measure as

$$\mathcal{W} = \mathcal{E}\left\{ \sum_{i\in B} \phi_i(t_i) - \sum_{i\in S} \psi_i(t_i)\right\} - \sum_i [1 - \bar{\alpha}_i(1)]V_i(1), \tag{32}$$

$$\text{where} \qquad \bar{\alpha}_i(t) = \mathcal{E}\{\, \alpha_i(t_i) \mid t_i \le t\,\}, \tag{33}$$

$$\phi_i(t) = u_i(t) + [1 - \bar{\alpha}_i(t)]tu_i'(t) \qquad \text{and} \qquad \psi_i(t) = v_i(t) + [1 - \bar{\alpha}_i(t)]tv_i'(t). \tag{34}$$

The functions ϕ_i and ψ_i for the buyers and sellers are called 'virtual valuations' by Myerson. In the expression for the welfare measure, it is important to note that the expectation is taken with respect to both the traders' types and the sets B and S of buyers and sellers who trade conditional on the entire vector (t_i) of types (feasibility requires that these two sets have equal cardinality). This representation shows that the design problem is summarized by choosing welfare weights, and for specified weights, choosing the rules of the mechanism to maximize the welfare measure via the induced sets B and S of successful traders. The requirement of 'individual rationality', in the form that each $V_i(t) \geq 0$, is satisfied by setting each $V_i(1) = 0$. Further, the mechanism maximizes the welfare measure if there exists an increasing function f such that the equilibrium induces trading sets B and S that maximize

$$\sum_{i \in B} f(\phi_i(t_i)) - \sum_{i \in S} f(\psi_i(t_i)) \tag{35}$$

subject to $|B| = |S|$ for each realization (t_i). Gresik and Satterthwaite (1989) show that rules for monetary payments exist that actually realize the equilibrium with these rules for trades of goods.

To take a special case, suppose there is a single seller with a commonly known valuation for a single item. The seller's optimal mechanism is obtained by setting the buyers' welfare weights to zero; consequently, it should sell the item to the buyer with the largest among the virtual valuations

$$\phi_i(t_i) = u_i(t_i) + t_i u_i'(t_i) = \bar{u}_i'(t_i) \tag{36}$$

provided it exceeds the seller's valuation. In the symmetric case $u_i \equiv u$, for example, this rule merely specifies that the buyer with the highest valuation should obtain the item if it is sold (since the actual and virtual valuations have the same ordering), and that the seller should use an optimal ask price. Thus, this mechanism conforms exactly to the usual auction formats. Bulow and Roberts (1989) observe that generally the virtual valuations are marginal revenues in the seller's calculations.

If the welfare weights are independent of the types then the auction design is *ex ante* efficient, and further, if they are all the same then the design maximizes the expected total surplus. An elaborate example of relevance to regulatory policy is worked out in detail by Riordan and Sappington (1987).

For the case of risk-averse buyers, Matthews (1983, 1984) and Maskin and Riley (1984) characterize the seller's design problem as an optimal control problem. They find that it is optimal for the seller to charge 'entry fees' that decline with the magnitude of the bid submitted (negative for large bids), and to reject the high bid with positive probability (though small if the bid is large). The essential idea is to impose risk on the buyers to motivate higher bids, but a buyer with a very high valuation is nearly perfectly insured (marginal utility differs little between winning and losing). With extremely risk averse buyers, the seller can attain nearly perfect price discrimination. Analogous results obtain in the case of risk-neutral bidders who have correlated private information, for example about an item of common value. Cremér and McLean (1985), McAfee, McMillan, and Reny (1989), and McAfee and Reny (1992) provide conditions under which the seller can in principle extract nearly all of the potential profit. In all cases, however, full exploitation of these features requires that the rules of the auction depend crucially on the probability distribution of buyers' information.

Border (1991) studies the 'reduced form' of an auction, interpreted as a function that assigns a probability of winning to each possible type of each bidder. He characterizes the set of all such functions that are implementable as auctions, and shows that this set can be represented as a convex polyhedron with extreme points that are associated with assignments that simply order the types. That is, the bidder whose type is ranked highest wins. This geometric characterization enables the implementation as an auction to be constructed from the solution to a linear programming problem.

Myerson and Satterthwaite (1983) examine the case of a single seller and a single buyer in which the welfare weights are identical constants, corresponding to *ex ante* incentive efficiency of the mechanism, as in Holmström and Myerson (1983). They note in the special case of valuations distributed uniformly on the same interval that an efficient mechanism is the static double auction in which the price is the average of the bid and offer submitted, assuming that the parties follow the linear equilibrium strategies identified by Chatterjee and Samuelson (1983) — although there are many nonlinear equilibria, only the linear one is efficient. Gresik and Satterthwaite (1989) construct *ex ante* efficient mechanisms for the general case of several sellers and several buyers with differing independent probability distributions of their valuations; cf. Gresik (1991c) for the case with correlated distributions. Their main result is that for *ex ante* efficient trading mechanisms the *ex post* inefficiency, as measured by the maximal difference between the valuations of a buyer and a seller who do not trade, is of order $\sqrt{\ln(M)}/M$ in terms of the minimum M of the numbers of buyers and sellers (as noted in §6 on double auctions, this bound is improved to $1/M$ by Satterthwaite and Williams (1989abc)). Generally, however, the rules of these efficient mechanisms depend on the distributions and therefore they do not conform to the usual forms of auctions; e.g., the payment rules they use (although they are not the only possible ones) often mandate payments by buyers who do not trade. Indeed, even for two sellers and two buyers with uniform distributions, the ordinary double auction that uses the price at the midpoint of the interval of clearing prices is inefficient. This difficulty motivates much of the work reported in §6 on double auctions, particularly those that demonstrate the efficiency of double auctions with many participants.

An alternative construction by Gresik (1991b) obtains stronger positive results. He strengthens the *interim* individual rationality constraint $V_i(t_i) \geq 0$ used above, to the *ex post* individual rationality constraint that each trader must obtain a nonnegative net profit in every contingency. In particular, participants who do not trade do not make or receive payments, and those who do trade make or receive payments bounded by their valuations. His main result establishes that there exists an open set of trading problems (in the space of probability distributions) for which the *ex ante* efficient mechanism can be implemented with payment rules that satisfy these stronger individual rationality constraints. This set is characterized by problems for which certain functions have unique roots, which he interprets as a 'single crossing property' of the sort assumed in many studies of incentive problems. The net result is the demonstration that mechanisms that enforce *ex post* rationality, and therefore conform more closely to standard auctions, but allow contingent selections of trading prices from the interval of clearing prices, are *ex ante* efficient in a nontrivial class of problems.

Similar methods can be applied to other contexts akin to auctions. We mention one among several examples in Kennan and Wilson (1991). Suppose that in a legal dispute a trial will cost each party c and yield a judgment $v = p - d$ paid to the plaintiff by the defendant, where initially the plaintiff knows p and the defendant knows d and they both know these have independent distribution functions F and G with densities f and g. Thus the gain from a pretrial settlement is $2c$. The incentive-compatible mechanism that maximizes the sum of the parties' *ex ante* expected payoffs can be derived using the methods above. One finds that they settle if

$$2c \geq \alpha \left[\frac{F(p)}{f(p)} + \frac{G(d)}{g(d)} \right] , \tag{37}$$

provided the right side is increasing, where α is a number chosen to ensure feasibility. In the case of uniform distributions, for example, if $c \leq 1/3$ then $\alpha = 2/3$ and they settle if $p + d \leq 3c$. Analogous to Myerson and Satterthwaite's example above, this optimal mechanism is implemented by a procedure in which the plaintiff asks P, the defendant offers D, and if $P < D$ then they settle on a payment $\frac{1}{2}[P + D]$, and otherwise go to trial. For this game the linear equilibrium has strategies $P = -2c + \frac{4}{3}p$ and $D = 2c + \frac{4}{3}d$. This procedure can also be used in a common-value model, although its optimality properties are unknown. Suppose (v, p, d) has a Normal distribution such that conditional on v, p and d are independent and identically distributed with mean v and variance $\frac{1}{2}s^2$, and consider the limiting case as the variance of the marginal distribution of v increases: at the limit, the linear equilibrium strategies are $P = \frac{1}{2}\Delta + p$ and $D = -\frac{1}{2}\Delta + d$, where $[2c/s]h(\Delta/s) = 1$ and $h = f/[1 - F]$ is the hazard function for the standard Normal distribution function. As in §3, similar results are obtained with a Lognormal distribution and trial costs that are proportional to the judgment.

11. Research Frontiers

Strategic analyses of auctions have developed rapidly, but significant gaps remain. The theory relies mainly on static formulations that invoke strong assumptions, such as symmetries among bidders, common knowledge of probability distributions, absence of risk aversion, *et cetera.* The fundamental assumption that an equilibrium predicts behaviors has rarely been relaxed. These assumptions facilitate theoretical work but they hamper empirical and experimental studies, since they are never precisely true in practice and little has been done to establish robustness of the predictions. Moreover, they thwart applications of the theory to practical affairs. Indeed, the paucity of reported applications and the occasional rejections of the theory (e.g., Levinson (1987)) by skilled practitioners indicate that more can be done to make it a useful tool.

Dynamic procedures, such as bid-ask markets, have received little attention although they have paramount importance in practice. The theory of efficient mechanisms remains a weak explanation for the prevalence of auction rules that are invariant to the characteristics of participants. Scant progress has been made in building theories with generality comparable to the Walrasian model of general equilibrium, even though the enigma of price formation in the Walrasian model is a prime motivation for studies of auctions.

Nevertheless, the methods of game theory have contributed substantially to the strategic analysis of auctions, and the main empirical studies [e.g., Hendricks and Porter (1988)] provide some support. This accomplishment stems partly from precise formulations and exact criteria for a solution, but most importantly it derives from explicit recognition of the effects of private information on strategic behavior. The emphasis on private information has brought game theory closer to practical affairs, and the resulting development of new techniques has enriched the methodology. One can hope that the emerging power of game theory to characterize market behavior will enable a general reformulation of economic models to include strategic behavior affected by private information.

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Table 1

Normal and Lognormal Bid Factors

n :	2	3	4	6	8	10	15	20
α_n :	1.772	1.507	1.507	1.595	1.686	1.763	1.909	2.014
s	$\beta_n(s)/B(s)$							
.01	.983	.985	.985	.984	.983	.983	.981	.980
.10	.847	.863	.861	.852	.844	.837	.824	.815
.25	.682	.698	.689	.667	.650	.636	.611	.594
.50	.498	.495	.474	.437	.411	.392	.360	.339
1.00	.281	.244	.212	.173	.150	.134	.111	.098

Table 2

Private and Common Factors Model

Bid Factor γ_n and Winner's Expected Profit %

			Bid Factor				Profit Percentage			
t^2	s^2	n :	2	4	8	16	2	4	8	16
.75	.25		.307	.314	.271	.228	37	17	9	6
.50	.50		.312	.366	.355	.329	47	27	17	12
.25	.75		.307	.410	.444	.451	59	40	30	23

Table 3

Bidding Strategies – One Perfectly Informed Bidder

n	:	1	2
α	:	.5000	.6667
β	:	.6796	.5349
Exp. Profit (Informed)	:	.0920	.0647
Exp. Profit(Estimator)	:	.0244	.0178
Exp. Revenue (Seller)	:	.3836	.3998

Table 4

Seller's Expected Revenue

α	:	.00	.25	.33	.50	.90	1.00
First-Price Auction	:	.3333	.3392	.3443	.3590	.4411	.5000
Second-Price Auction	:	.3333	.3290	.3125	.2962	.2585	.2500