Stability Properties of the Rate-of-Return Regulation Process

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Abstract: This note provides proofs for several of the claims made in our paper “Dynamics of Rate-of-Return Regulation,” referenced from heron as Nezlobin, Rajan and Reichelstein (2011).
The following two proofs provide details for the claims in Substeps B1 and C1 of Proposition 2 in Nezlobin, Rajan and Reichelstein (2011).

**Sub-Step B1.** If state $\theta'_t$ is feasible for some $t$, $P_t (K (\theta'_t)) = p_\infty$, and there exists a subsequence of $s(t)$, $s(t_i)$, such that

\[ \theta'_t = (1 + \mu)^t \lim_{i \to \infty} s(t_i). \]  

(1)

then there exists an investment $I'_t$ such that state $\theta'_{t+1}$ is feasible,

\[ P_{t+1} (K (\theta'_{t+1})) = p_\infty, \]

and there exists a subsequence of $s(t)$, $s(t'_i)$, such that

\[ \theta'_{t+1} = (1 + \mu)^{t+1} \lim_{i \to \infty} s(t'_i). \]  

(2)

**Proof of Sub-Step B1**

Consider a sequence $\{s(t_i + 1)\}_{i=1}^\infty$ which is comprised of elements of $\{s(t)\}$ immediately following the elements of $\{s(t_i)\}$ from (1). The sequence $\{s(t_i + 1)\}_{i=1}^\infty$ must have a converging subsequence (since all vectors $s(\cdot)$ are drawn from a compact set). Let $\{s(t'_i)\}_{i=1}^\infty$ denote this converging subsequence and let

\[ \varphi = (1 + \mu)^{t+1} \lim_{i \to \infty} s(t'_i). \]

To show that $\varphi$ is a legitimate candidate for $\theta'_{t+1}$, we need to check that the first $T - 1$ elements of $\varphi$ correspond to the last $T - 1$ elements of $\theta'_t$. Letting $\varphi_\tau$ denote the $\tau$-th coordinate of $\varphi$, we have:

\[ \varphi_\tau = (1 + \mu)^{t+1} \lim_{i \to \infty} s_\tau(t'_i). \]

By construction, for $\tau < T$

\[ s_\tau(t'_i) = (1 + \mu)^{-1} s_{\tau+1}(t'_i - 1). \]

Therefore, for $\tau < T$,

\[ \varphi_\tau = (1 + \mu)^{t+1} \lim_{i \to \infty} s_\tau(t'_i) = (1 + \mu)^t \lim_{i \to \infty} s_{\tau+1}(t'_i - 1) = (1 + \mu)^t \lim_{i \to \infty} s_{\tau+1}(t_i) = I'_{t-T+\tau}. \]

Setting $I'_t = \varphi_T$, we will have

\[ \theta'_{t+1} = \varphi = (1 + \mu)^{t+1} \lim_{i \to \infty} s(t'_i). \]
It remains to check that $\theta'_{t+1}$ is feasible and that $P_{t+1}(K(\theta'_{t+1})) = p_\infty$. To that end, we apply the assumption of constant market growth as well as the linearity of $K(\cdot)$ and $C(\cdot)$:

$$K(\theta'_{t+1}) = (1 + \mu)^{t+1} K_\infty,$$

$$P_{t+1}(K(\theta'_{t+1})) K(\theta'_{t+1}) = (1 + \mu)^{t+1} P_{t+1}((1 + \mu)^{t+1} K_\infty) K_\infty = (1 + \mu)^{t+1} p_\infty K_\infty,$$

$$C(\theta'_{t+1}) = (1 + \mu)^{t+1} p_\infty K_\infty.$$

---

**Sub-Step C1.** *The sequence $\{s_t\}$ converges to some $s^*$.*

**Proof of Sub-Step C1.** We know that:

$$s_t = \frac{x_1 - x_2}{x_1} s_{t-1} + \ldots + \frac{x_T - x_{T+1}}{x_1} s_{t-T}. \quad (3)$$

All coefficients in the right-hand-side are strictly positive and add up to unity. Let

$$\overline{s}_t = \max_{\tau=0,...,T-1} s_{t-\tau}$$

and

$$\underline{s}_t = \min_{\tau=0,...,T-1} s_{t-\tau}.$$

Equation (3) implies that $\underline{s}_{t-1} \leq s_t \leq \overline{s}_{t-1}$. Therefore, it has to be that $\underline{s}_{t-1} \leq s_t \leq \overline{s}_t \leq \overline{s}_{t-1}$. These inequalities imply that both sequences $\{\underline{s}_t\}$ and $\{\overline{s}_t\}$ are monotonic and must converge to some limits, $\underline{s}^*$ and $\overline{s}^*$, respectively. Let us assume that $\underline{s}^* \neq \overline{s}^*$. Then, by equation (3),

$$s_t \geq \underline{s}_{t-1} + \left( \min_{\tau=1..T} \frac{x_\tau - x_{\tau+1}}{x_1} \right) \left( \overline{s}_{t-1} - \underline{s}_{t-1} \right).$$

For sufficiently large values of $t$, this becomes

$$s_t \geq \underline{s}^* + \left( \min_{\tau=1..T} \frac{x_\tau - x_{\tau+1}}{x_1} \right) \left( \overline{s}^* - \underline{s}^* \right) - \epsilon.$$

Therefore, for large values of $t$,

$$\underline{s}_t \geq \underline{s}^* + \left( \min_{\tau=1..T} \frac{x_\tau - x_{\tau+1}}{x_1} \right) \left( \overline{s}^* - \underline{s}^* \right) - \epsilon.$$

On the other hand, $\underline{s}_t$ converges to $\underline{s}^*$, so for large $t$’s, $\underline{s}_t$ has to be not greater than $\underline{s}^* + \epsilon$. Choosing $\epsilon$ so that

$$\underline{s}^* + \epsilon < \underline{s}^* + \left( \min_{\tau=1..T} \frac{x_\tau - x_{\tau+1}}{x_1} \right) \left( \overline{s}^* - \underline{s}^* \right) - \epsilon$$

yields a contradiction. We have shown that $\underline{s}^* = \overline{s}^* \equiv s^*$ and, since $\underline{s}_t \leq s_t \leq \overline{s}_t$, $s_t$ converges to $s^*$. 

$\blacksquare$
The following result states and proves Proposition 5 in Nezlobin, Rajan and Reichelstein (2011).

**Proposition A** Given one-hoss shay productivity, straight-line depreciation and stationary demand, the trajectory $\theta^*$ provides a locally stable implementation of $p^*$.

**Proof of Proposition A**

Consider the equilibrium trajectory, $\theta^*$, and some other trajectory, $\theta$, such $\theta_T$ is close to $\theta^*_T$. Let $\varepsilon_t = I_t - I^*_t$ and $\varepsilon_t = \theta_t - \theta^*_t = (\varepsilon_{t-T}, \ldots, \varepsilon_{t-1})$ denote the difference between the two investment paths. We will consider the mapping, $\Phi$, which defines transitions from vector $\varepsilon_t$ to vector $\varepsilon_{t+1}$. The proof consists of three major steps. First, it will be shown that, when $\varepsilon_t$ is sufficiently small, the mapping $\Phi$ is uniquely defined and that it is constant over time. In the second step, this mapping will be linearized around the zero-vector. The proof will be concluded by showing that the zero-vector is an attractor for the linearized mapping.

**Step 1:** $\Phi$ is well-defined.

We first show that the mapping $\Phi$ is constant over time. Let $\varepsilon^i_t$ denote the $i$-th component of $\varepsilon_t$. Then,

$$
\varepsilon^i_t = \varepsilon_{t-1-T+i} = I_{t-1-T+i} - I^*_{t-1-T+i}.
$$

Therefore, for the first $T-1$ components of $\varepsilon_{t+1} = \Phi(\varepsilon_t)$, we have:

$$
\varepsilon^i_{t+1} = \varepsilon^{i+1}_{t}.
$$

(4)

This part of the mapping clearly does not depend on $t$. The last component of $\varepsilon_{t+1}$ is:

$$
\varepsilon^T_{t+1} = I_t - I^*_t.
$$

Since there is no growth in the output market, the investments on the equilibrium path are constant over time and so is the vector $\theta^*_t$. Investment $I_t$ is chosen so as to satisfy the RoR constraint:

$$
R_{t+1}(K(\theta_{t+1})) = C(\theta_{t+1}).
$$

(5)

We also know that

$$
R_{t+1}(K(\theta^*_{t+1})) = C(\theta^*_{t+1}).
$$

(6)

Subtracting (6) from (5), one obtains:

$$
R_{t+1}(K(\theta_{t+1})) - R_{t+1}(K(\theta^*_{t+1})) = C(\varepsilon_{t+1}).
$$

(7)

As demand is constant over time, this is equivalent to:

$$
R_0(K(\theta_{t+1})) - R_0(K(\theta^*_{t+1})) = C(\varepsilon_{t+1}),
$$

3
or,

$$R_0(K(\varepsilon_{t+1} + \theta^*_{t+1})) - R_0(K(\theta^*_{t+1})) = C(\varepsilon_{t+1}). \quad (8)$$

The last component of $\varepsilon_{t+1}$, $\varepsilon^T_{t+1}$, is defined by the equation above. This equation does not depend on $t$ (since $\theta^*_{t+1}$ is constant over time), therefore $\Phi$ is constant over time.

To check that $\Phi$ is uniquely defined when $\varepsilon_t$ is close to the zero vector, we need to show that equation (8) defines $\varepsilon^T_{t+1}$ uniquely in that case. Applying the assumption that assets have one-hoss shay productivity, (8) can be expanded in the following way:

$$R_0(\varepsilon^T_{t+1} + \varepsilon^T_t + \ldots + \varepsilon^2_t + K^*) - R_0(K^*) - (w + z^0_1)\varepsilon^T_{t+1} - (w + z^0_2)\varepsilon^T_t - \ldots - (w + z^0_T)\varepsilon^2_t = 0, \quad (9)$$

where $K^* = K(\theta^*_{t+1})$. Let us now apply the implicit function theorem to show that, in the neighborhood of the zero-vector, there exists a continuous function $\varepsilon^T_{t+1}(\varepsilon_t)$ satisfying (8), such that $\varepsilon^T_{t+1}(0) = 0$. According to this theorem, we need to check that the partial derivative of the left-hand-side of (9) with respect to $\varepsilon^T_{t+1}$ is different from zero. This derivative is equal to:

$$R_0'(\varepsilon^T_{t+1} + \varepsilon^T_t + \ldots + \varepsilon^2_t + K^*) - w - z^0_1.$$

At $\varepsilon_t = 0$ and $\varepsilon^T_{t+1} = 0$, this expression reduces to:

$$R_0'(K^*) - w - z^0_1.$$

Note that

$$R_0'(K^*) = p^* + P_0'(K^*) < p^* < w + z^0_1.$$

Therefore, the partial derivative of the LHS of (9) with respect to $\varepsilon^T_{t+1}$ is negative, and there exists a solution to (8), $\varepsilon^T_{t+1}(\varepsilon_t)$, which is close to 0 when $\varepsilon_t$ is close to 0.

Let us now show that this solution is unique. Assume that there exists some other $\hat{\varepsilon}^T_{t+1}$ also satisfying (9). Let $\Delta = \hat{\varepsilon}^T_{t+1} - \varepsilon^T_{t+1}$. Then, subtracting two versions of (8) from one another, we have:

$$R_0(K(\varepsilon_{t+1} + \theta^*_{t+1}) + \Delta) - R_0(K(\varepsilon_{t+1} + \theta^*_{t+1})) = \Delta \cdot (w + z^0_1).$$

Since $R_0(\cdot)$ is concave, $R_0(0) = 0$, and both $K(\varepsilon_{t+1} + \theta^*_{t+1} + \Delta)$ and $K(\varepsilon_{t+1} + \theta^*_{t+1})$ are greater than 0, it follows that

$$\frac{R_0(K(\varepsilon_{t+1} + \theta^*_{t+1} + \Delta) - R_0(K(\varepsilon_{t+1} + \theta^*_{t+1}))}{\Delta} < \frac{R_0(K(\varepsilon_{t+1} + \theta^*_{t+1}))}{K(\varepsilon_{t+1} + \theta^*_{t+1})} = P_0(K(\varepsilon_{t+1} + \theta^*_{t+1})).$$

Now recall that we defined $\varepsilon^T_{t+1}$ to be the root close to zero, so

$$P_0(K(\varepsilon_{t+1} + \theta^*_{t+1})) \approx p^* < w + z^0_1.$$
Therefore,
\[ R_0(K(\varepsilon_{t+1} + \theta^*_t) + \Delta) - R_0(K(\varepsilon_{t+1} + \theta^*_t)) < \Delta \cdot (w + z_1^o) \]
and there exists a neighborhood of \(0\) where \(\Phi\) is uniquely defined.

**Step 2: Linearization of \(\Phi\).**

As was shown in equation (4), for \(1 \leq i \leq T - 1\)

\[ \varepsilon_{t+1}^i = \varepsilon_t^{i+1}. \]

The last component of \(\varepsilon_{t+1}, \varepsilon_{t+1}^T\) is defined by equation (9). To linearize \(\Phi\) around \(0\), we will apply the first-order expansion to \(\varepsilon_{t+1}^T\). As \(\varepsilon_{t+1}^T(0) = 0\),

\[ \varepsilon_{t+1}^T \approx \frac{\partial \varepsilon_{t+1}^T}{\partial \varepsilon_t} \cdot \varepsilon_t^1 + \ldots + \frac{\partial \varepsilon_{t+1}^T}{\partial \varepsilon_t^T} \cdot \varepsilon_t^T. \]

The partial derivatives can be computed from equation (9) using the implicit function differentiation theorem. At \(\varepsilon_t = 0\) they are equal to:

\[ \frac{\partial \varepsilon_{t+1}^T}{\partial \varepsilon_t^i} = \frac{w + z_2^o - R_0'(K^*)}{R_0'(K^*) - w - z_1^o} \]

for \(2 \leq i \leq T\) and \(\frac{\partial \varepsilon_{t+1}^T}{\partial \varepsilon_t^1} = 0\).

For brevity, let us write \(R'\) for \(R_0'(K^*)\). For future reference, we note that \(R_0'(K^*) < P_0(K^*) = p^*\). The linearized mapping takes the following form:

\[ \begin{pmatrix} \varepsilon_{t+1}^1 \\ \varepsilon_{t+1}^2 \\ \vdots \\ \varepsilon_{t+1}^T \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \ldots & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 1 & \ldots \\ 0 & \frac{w + z_2^o - R'}{R' - w - z_1^o} & \ldots & \frac{w + z_2^o - R'}{R' - w - z_1^o} & \ldots & \frac{w + z_2^o - R'}{R' - w - z_1^o} \end{pmatrix} \begin{pmatrix} \varepsilon_t^1 \\ \varepsilon_t^2 \\ \vdots \\ \varepsilon_t^T \end{pmatrix} \]

(10)

**Step 3: Proof of Stability.**

To show that the zero-vector is an attractor for the linear system (10), we will apply the Jury stability criterion. The first step is to compute the characteristic polynomial of this system. Let \(a_0, \ldots, a_T\) be the coefficients of this polynomial, \(F(u)\), on degrees \(T, \ldots, 0\), respectively, with \(a_0\) normalized to unity. Given the form of the transition matrix in (10), one can see that:

\[ a_0 = 1, \]
\[ a_1 = -\frac{w + z_2^o - R'}{R' - w - z_1^o}, \]
\[ \ldots \]
\[ a_t = -\frac{w + z_{i+1}^o - R'}{R' - w - z_1^o}, \]
\[ a_{T-1} = -\frac{w + z_T^o - R'}{R' - w - z_1^o}, \]
\[ a_T = 0. \]

Jury’s criterion states that zero-vector is an attractor for a linear system if all the roots of the characteristic polynomial, corresponding to that system, lie in the zero circle. The Jury test is a procedure for checking that this condition holds for a specific polynomial. To use this test, we first construct the following array:

\[
\begin{array}{cccccc}
  a_T^{(0)} & a_T^{(0)} & \cdots & a_1^{(0)} & a_0^{(0)} \\
  a_0^{(0)} & a_1^{(0)} & \cdots & a_{T-1}^{(0)} & a_T^{(0)} \\
  a_{T-1}^{(1)} & a_{T-2}^{(1)} & \cdots & a_0^{(1)} \\
  a_0^{(1)} & a_1^{(1)} & \cdots & a_{T-1}^{(1)} \\
  a_{(T-2)}^{(T-2)} & a_{(T-2)}^{(T-2)} & a_1^{(T-2)} & a_2^{(T-2)} \\
  a_0^{(T-2)} & a_1^{(T-2)} & a_0^{(T-2)} & a_T^{(T-2)} \\
\end{array}
\]

In this array \( a_t^{(0)} = a_t \) for all \( t \), and, then, each next odd row is computed as follows:

\[
\begin{align*}
  a_t^{(i)} &= \begin{vmatrix}
    a_{t+1}^{(i-1)} & a_0^{(i-1)} \\
    a_{T-1-t}^{(i-1)} & a_{T+1}^{(i-1)}
  \end{vmatrix} \\
\end{align*}
\]  \( (11) \)

for 0 \( \leq t \leq T - i \). Four conditions have to be met, for the polynomial \( F(u) \) to have all the roots inside the unit circle. First, \( |a_T^{(0)}| \) must not exceed \( a_0^{(0)} \). This condition is obviously satisfied as \( a_T^{(0)} = 0 \) and \( a_0^{(0)} = 1 \). Second, we need to check that \( F(1) \) is greater than zero. Third, \( F(-1) \) must be greater than zero for even \( T \) and less than zero for odd \( T \). Finally, for each \( i \geq 1 \), \( |a_{T-i}^{(i)}| \) must be greater than \( |a_0^{(i)}| \). In what follows, we check these conditions sequentially.

First let us compute \( F(1) \). Using expressions for coefficients obtained earlier, we get:

\[ F(1) = 1 - \sum_{i=1}^{T-1} \frac{(w + z_{i+1}^o - R')}{(R' - w - z_1^o)}. \]

Since, as was noted earlier, \( R' - w - z_1 < 0 \), we need to show that

\[ (R' - w - z_1) - \sum_{i=1}^{T-1} (w + z_{i+1}^o - R') < 0, \]
or, that
\[
\sum_{i=0}^{T-1} (w + z_{i+1}^o - R') > 0. \tag{12}
\]

Note that \( R' < p^* \), which, under the assumptions made, is equal to:
\[
p^* = w + \frac{\sum_{i=1}^{T} z_i^o}{T}.
\]

Therefore,
\[
TR' < \sum_{i=1}^{T} (w + z_i^o)
\]
and condition (12) holds.

Let us now check that \( F(-1) \) is greater than zero for even \( T \) and less than zero for odd \( T \). First, note that \( F(-1) \) has the same sign as:
\[
(w + z_1^o - R')F(-1) = (w + z_1^o - R')(-1)^T + ... + (w + z_T^o - R')(-1)^1. \tag{13}
\]

Since the firm uses straight-line depreciation to account for its assets, we have:
\[
z_i^o = v \left( \frac{1}{T} + \frac{T + 1 - i}{T} r \right).
\]

Therefore, \( z_i^o \) form an arithmetic progression.

By equation (13), for even values of \( T \), we have:
\[
(w + z_1^o - R')F(-1) = v \cdot \frac{T}{2} \cdot \frac{1}{T} r = vr/2 > 0.
\]

For odd values of \( T \), the corresponding expression is:
\[
(w + z_1^o - R')F(-1) = -(w + (\frac{1}{T} + r)v - R') + \frac{T-1}{2} \cdot \frac{1}{T} rv = R' - w - \frac{1}{T} v - \frac{T+1}{2} rv.
\]

However, we know that
\[
R' < p^* = w + \frac{1}{T} v + \frac{T(T + 1)}{2T} rv = w + \frac{1}{T} v + \frac{T + 1}{2} r.
\]

Therefore,
\[
(w + z_1^o - R')F(-1) < 0.
\]

It remains to show that for each \( i \geq 1 \),
\[
|a_{T-i}^{(i)}| > |a_0^{(i)}|.
\]
Let $b$ denote the following quantity:

$$b = \frac{1}{T}rv \frac{z_1^o + w - R'}{z_1^o + w - R'}.$$

Then, the coefficients of the characteristic polynomial can be rewritten as:

$$a_t^{(0)} = 1 - bt$$

for $0 \leq t \leq T - 1$ and $a_T^{(0)} = 0$.

In the rest of the proof, the following upper-bound on $b$ will be employed. Observe that

$$R' < p^* = w + \frac{z_1^o + \ldots + z_T^o}{T} = w + \frac{z_1^o + z_T^o}{2}.$$

After simple algebra, the inequality above translates into:

$$b(T - 1) = \frac{(T - 1)rv}{T(z_1^o + w - R')} < 2.$$

Let us now apply formula (11) to compute the third row in the Jury array:

$$a_t^{(1)} = \begin{vmatrix} a_{t+1}^{(0)} & 1 \\ a_{T-1-t}^{(0)} & 0 \end{vmatrix} = -a_{T-1-t}^{(0)} = (T - 1 - t)b - 1.$$

Therefore, $|a_{T-1}^{(1)}| = 0$. On the other hand,

$$|a_0^{(1)}| = |(T - 1)b - 1| < 1$$

by the upper-bound on $b$ derived earlier. We have shown that $|a_0^{(1)}| < |a_{T-1}^{(1)}|$.

The fifth row of the array can be computed as follows:

$$a_t^{(2)} = \begin{vmatrix} a_{t+1}^{(1)} & a_0^{(1)} \\ a_{T-2-t}^{(1)} & a_{T-1}^{(1)} \end{vmatrix} = \begin{vmatrix} (T - 2 - t)b - 1 & (T - 1)b - 1 \\ (t + 1)b - 1 & (t + 1)b - 1 \end{vmatrix}.$$

The determinant above reduces to:

$$a_t^{(2)} = (2 - (T - 1)b)T \cdot (t + 1) = \delta \cdot (t + 1)$$

where $\delta = (2 - (T - 1)b)T > 0$. Clearly, $|a_0^{(2)}| < |a_{T-2}^{(2)}|$. Let us now show that all subsequent odd rows will have the same structure, namely that, if $a_t^{(i-1)} = \delta(t + 1)$ for some positive $\delta$, then

$$a_t^{(i)} = \delta'(t + 1)$$

for some positive $\delta'$. Indeed,

$$a_t^{(i)} = \begin{vmatrix} (t + 2)\delta & \delta \\ (T - i - t + 1)\delta & (T - i + 2)\delta \end{vmatrix} = \delta^2(T + 3 - i)(t + 1). \tag{14}$$

All remaining inequalities follow immediately.
The following result states a variant of Proposition 6 in Nezlobin, Rajan and Reichelstein (2011). The result assumes the geometric setting and furthermore assumes that the intercept term of inverse demand functions, \( a \), is sufficiently large. In particular, in the scenario with stationary markets (\( \mu = 0 \)), it will be assumed that:

\[
a \geq \max \left\{ \begin{array}{l}
w + v (r + \delta) \frac{(\alpha(1-\delta)+\sqrt{1-\delta}(\alpha-\delta))}{\delta(1-\delta)} \\
w + v (r + \delta) \frac{2\alpha^2-\delta}{\alpha^2+\delta-2a\delta}, \\
w + v (r + \delta) \frac{\alpha^2-\delta^2}{(1-\delta)\delta} \end{array} \right\}.
\]

In the constant growth scenario (when all \( \mu_t \) are equal), the inequalities above are assumed to be satisfied for

\[
\alpha' = \frac{\alpha + \mu}{1 + \mu} \quad (15)
\]

and

\[
\delta' = \frac{\delta + \mu}{1 + \mu} \quad (16)
\]

**Proposition B** Suppose that depreciation is more decelerated than the R.P.C. rule (\( \alpha > \delta \)) and the firm starts from a feasible state. Assume further that i) \( P(0) \) is sufficiently large and ii) if two possible \( I_t \) exist that result in a feasible \( \theta_{t+1} \), the firm chooses the larger one. Then,

\[ P(K(\theta_t)) \to p^*. \]

**Proof of Proposition B**:

First, observe that using the Auxiliary Lemma in the proof of Lemma 1 in Nezlobin, Rajan and Reichelstein (2011), the model with constant growth can be converted to the model with stationary markets with parameters \( \alpha' \) and \( \delta' \) given by equations (15) and (16). Therefore, it is sufficient to establish the result for the case of stationary markets.

Now let us show that if the firm is in a feasible state in period \( t \), there will exist at least one \( I_t \geq 0 \), such that the firm will be in a feasible state in period \( t + 1 \). Following the first steps of the proof of Lemma 1 in Nezlobin, Rajan and Reichelstein (2011), one can identify the following two candidates for \( K_{t+1} \):

\[
K_{t+1}^{(1)} = \frac{1}{2b} (a - w - v(r + \delta)) + \frac{1}{2b} \sqrt{S(K_t)} \quad (17)
\]

and

\[
K_{t+1}^{(2)} = \frac{1}{2b} (a - w - v(r + \delta)) - \frac{1}{2b} \sqrt{S(K_t)}, \quad (18)
\]

where

\[
S(K_t) \equiv (a - w - v(r + \delta))^2 + 4b(-a + rv + w - rv\alpha + (a + v - w - v\alpha)\delta)K_t + 4b^2(1 - \delta)K_t^2. \quad (19)
\]
To prove that $K_{t+1}^{(1)}$ is a real number, one needs to verify that $S(K_t)$, as given by equation (19), is non-negative. Note that $S(K_t)$ is quadratic in $K_t$ with a positive coefficient on $K_t^2$. Therefore, this function will be always non-negative whenever the discriminant is negative.

The discriminant of $S(K_t)$ is:

$$b^2 (w + r(v - v\alpha) - a (1 - \delta) - (w - v (1 - \alpha)) \delta)^2 - (1 - \delta) (a - w - v (r + \delta))^2.$$  

This is a quadratic function in $a$ with a negative coefficient on $a^2 ((1 - \delta)^2 - (1 - \delta))$. The discriminant is zero for the following values of $a$:

$$w + v \frac{(r + \delta)}{\delta (1 - \delta)} \left( \alpha (1 - \delta) \pm \sqrt{1 - \delta} (\alpha - \delta) \right).$$

Therefore, $K_{t+1}^{(1)}$ is a real number, provided that

$$a > w + v \frac{(r + \delta)}{\delta (1 - \delta)} \left( \alpha (1 - \delta) + \sqrt{1 - \delta} (\alpha - \delta) \right). \quad (20)$$

Now let us check that $I_t^{(1)}$ identified in the proof of Lemma 1 in Nezlobin, Rajan and Reichelstein (2011) is positive. Recall that

$$I_t^{(1)} = \frac{1}{2b} (a - w - v (r + \delta) - 2 (1 - \alpha) b K_t) + \frac{1}{2b} \sqrt{S(K_t)}.$$  

For $I_t^{(1)}$ to be negative it has to be that:

1) $K_t > \frac{a - w - v (r + \delta)}{2 (1 - \alpha) b}$

and

2) $S(K_t) < (a - w - v (r + \delta) - 2 (1 - \alpha) b K_t)^2$.

Condition ii) is equivalent to

$$-K_t \left( (a - w) (\alpha - \delta) - b (2\alpha - \alpha^2 - \delta) K_t \right) < 0.$$  

Therefore, condition ii) is satisfied for

$$0 \leq K_t \leq \frac{(a - w) (\alpha - \delta)}{b(2\alpha - \alpha^2 - \delta)}.$$  

Conditions i) and ii) cannot be satisfied simultaneously if

$$\frac{(a - w) (\alpha - \delta)}{b(2\alpha - \alpha^2 - \delta)} < \frac{a - w - v (r + \delta)}{2 (1 - \alpha) b}.$$
Note that
\[ \frac{1}{2 (1 - \alpha)} > \frac{(\alpha - \delta)}{2\alpha - \alpha^2 - \delta}. \]

Therefore, conditions i) and ii) cannot be satisfied if
\[ a > w + v (r + \delta) \frac{(2\alpha - \alpha^2 - \delta)}{\alpha^2 + \delta - 2\alpha\delta}. \]  

When (20) and (21) are satisfied, \( I^{(1)}_t \) exists and is positive. Since the firm is assumed to choose the largest feasible investment and \( I^{(1)}_t > I^{(2)}_t \), we know that \( I^{(1)}_t \) will be chosen and \( K^{(1)}_{t+1} \) will be the capacity available in period \( t + 1 \). It remains to show that prices will converge to \( p^* \) if the trajectory is continued recursively, according to the definition of \( K^{(1)}_{t+1} \).

Exactly as in the proof of Proposition 6, it can be shown that \( K_{t+1} > K_t \) if \( K_t < K^*_t \) and \( K_{t+1} < K_t \) otherwise. Let us now show that \( K_t \) converge. Assume first that
\[ \frac{1}{2b} (a - w - v (r + \delta)) \geq K^*. \]

Starting from \( K_2 \),
\[ K_t > \frac{1}{2b} (a - w - v (r + \delta)) \geq K^*. \]

Since \( K_t \geq K^* \), we know that \( K_{t+1} \leq K_t \). Therefore, the sequence converges. Now assume that
\[ \frac{1}{2b} (a - w - v (r + \delta)) < K^*. \]

We will show that \( K_{t+1} > K^* \) if \( K_t > K^* \). This will imply that \( K_t \) converge once \( K_{t_0} > K^* \) for some \( t_0 \). If \( K_t \leq K^* \) for all \( t \), then the sequence must converge as well since \( K_t \) will be increasing in that area. From (22) it follows that \( K_{t+1} > K^* \) if
\[ \left( a - w - v (r + \delta) - 2\frac{(a - w) \delta - v\alpha (r + \delta)}{\delta} \right)^2 < S(K_t). \]

The inequality above is equivalent to:
\[ \frac{4 ((w - a) \delta + v\alpha (r + \delta) + bK_t \delta) (v(\alpha - \delta)(r + \delta) - b(1 - \delta)\delta K_t)}{\delta^2} < 0. \]  

The quadratic function in the numerator of the left-hand-side has two roots:
\[ \left\{ K^*, \frac{v(\alpha - \delta)(r + \delta)}{b(1 - \delta)} \right\}. \]

To show that condition (23) holds for \( K_t > K^* \), it suffices to check that
\[ \frac{v(\alpha - \delta)(r + \delta)}{b(1 - \delta)} < K^* = \frac{(a - w) \delta - v\alpha (r + \delta)}{b\delta}. \]
This inequality is satisfied if

\[ a > w + \frac{v (r + \delta) (\alpha - \delta^2)}{(1 - \delta) \delta}. \]

We have shown that when \( a \) is sufficiently large, capacity levels \( K_t \) converge. To conclude the proof, it remains to show that prices must converge as well. This last step is equivalent to the last step of the proof of Proposition 6 in Nezlobin, Rajan and Reichelstein (2011). ■