

# Information Aggregation in Dynamic Markets with Strategic Traders

## Technical Appendix

This appendix presents a discretized model of trading based on the model of Kyle (1985). It shows that (i) an equilibrium in this model always exists and (ii) information about separable securities always gets aggregated.

### 1 Model

There are  $I$  players,  $i = 1, \dots, I$ . There is a finite set of states of the world,  $\Omega$ , and a random variable (“security”)  $X : \Omega \rightarrow \mathbb{R}$ . Each player  $i$  receives information about the true state of the world,  $\omega \in \Omega$ , according to partition  $\Pi_i$  of  $\Omega$ . The join of partitions  $\Pi_1, \dots, \Pi_I$  consists of singleton sets.  $\Pi = (\Pi_1, \dots, \Pi_n)$  is the *partition structure*. Players have a common prior distribution  $P$  over states in  $\Omega$ .

Trading is organized as follows. At time  $t_0 = 0$ , nature draws a state,  $\omega^*$ , according to  $P$ , and all strategic players  $i$  observe their information  $\Pi_i(\omega^*)$ . At time  $t_1 > 0$ , each strategic player  $i$  chooses his demand  $d_1^i$  from  $2M_1 + 1$  possible actions  $\{-M_1d_1, -(M_1 - 1)d_1, \dots, M_1d_1\}$ , where unit of discretization  $d_1$  is a positive real number and bound  $M_1$  is a positive integer. At the same time, there is demand  $u_1$  from noise traders, drawn randomly from a “discretized” normal distribution with mean zero, “variance parameter”  $\sigma_1^2 > 0$ , and the same unit of discretization  $d_1$ . Formally, random variable  $\tilde{u}_1$  is drawn from  $N(0, \sigma_1^2)$ , and then  $u_1$  is equal to  $\tilde{u}_1$  rounded to the nearest multiple of  $d_1$  (i.e., if  $\tilde{u}_1 \in [-0.5d_1, 0.5d_1)$ , then  $u_1 = 0$ ; if  $\tilde{u}_1 \in [0.5d_1, 1.5d_1)$ , then  $u_1 = d_1$ ; and so on). Competitive market makers observe aggregate demand  $v_1 = \sum_i d_1^i + u_1$ , form their posterior beliefs about the true state of the world, and set market price  $y_1$  equal to the expected value of the security conditional on that posterior belief. The market clears, and all traders observe price  $y_1$  and aggregate volume  $v_1$ .

At time  $t_2 > t_1$ , the next auction takes place, with each strategic player  $i$  choosing demand  $d_2^i$  from a set of  $2M_2 + 1$  possible actions, with the unit of discretization  $d_2$ , and with demand from noise traders,  $u_2$ , drawn randomly from the discretized normal distribution with mean zero, variance parameter  $\sigma_2^2$ , and unit of discretization  $d_2$ . Subsequently, auctions are held at times  $t_k$ , with demand from noise traders drawn from the discretized normal distribution with variance parameter  $\sigma_k^2$  and unit of discretization  $d_k$ , either until the last auction, which takes place at time  $t_K \leq 1$  (in the case of finite games), or for an infinite number of periods  $t_k \leq 1$  (in the case of infinite games; in that case we say  $K = \infty$ ). The value of the security,  $x^* = X(\omega^*)$ , is revealed at some time  $t^* > 1$ . Trader  $i$ 's payoff is equal to  $\sum_{k=1}^K d_k^i (x^* - y_k)$ .

Strategy  $S_i$  of player  $i$  is a set of functions  $s_{ik}(\pi, d_1^i, \dots, d_{k-1}^i, v_1, \dots, v_{k-1})$  with values in the  $2M_k$ -simplex, denoting the probability distribution over the  $2M_k + 1$  possible actions of player  $i$  in period  $k$  after he observed partition  $\pi$ , submitted demands  $d_1^i, \dots, d_{k-1}^i$  in the first  $k-1$  periods, and

observed aggregate demands  $v_1, \dots, v_{k-1}$  in those periods. Strategy profile  $S$  is a set of strategies of individual players.

The pricing rule  $Y$  corresponding to strategy profile  $S$  is a set of functions  $y_k(v_1, \dots, v_k)$  such that  $y_k(v_1, \dots, v_k)$  is equal to the conditional expectation of the value of security  $X$  given that the true state was drawn according to prior probability  $P$ , each strategic trader  $i$  played according to strategy  $S_i$ , and the observed aggregate demands in the first  $k$  periods were  $v_1, \dots, v_k$ .

The resulting trading game is denoted  $\Gamma$ .

**Definition A.1** *Let  $Y$  be the pricing rule corresponding to strategy profile  $S$ . Strategy profile  $S$  is a perfect Bayesian equilibrium of game  $\Gamma$  if for every player  $i$ , every period  $k \leq K$ , and every alternative strategy  $S'_i$ , the expected continuation payoff of player  $i$ ,  $\sum_{k'=k}^K d_{k'}^i(x^* - y_{k'})$ , is at least as high under strategy  $S_i$  as it is under strategy  $S'_i$ , given that other traders  $j \neq i$  follow strategies  $S_j$  and the pricing rule is given by  $Y$ .*

The analysis (and the definition of equilibrium) are simplified by the fact that the traders and market makers can never observe a zero-probability event (unless they themselves deviated from their prescribed equilibrium strategy); i.e., we do not need to worry about out-of-equilibrium beliefs.

## 2 Finite games

The first result shows that an equilibrium of any finite game  $\Gamma$  always exists.

**Theorem A.1** *There exists a perfect Bayesian equilibrium  $S^*$  of any trading game  $\Gamma$  with  $K < \infty$ .*

**Proof.** We first introduce some notation. Take any  $n$  and let  $F^n$  denote the set of bounded functions  $f : \mathbb{Z}^n \rightarrow \mathbb{R}$ . For any function  $f \in F^n$ , define its norm  $\|f\|$  as  $\sum_{x=(x_1, \dots, x_n) \in \mathbb{Z}^n} |f(x)| \frac{1}{2^{|x_1| + \dots + |x_n|}}$ . Define the distance between functions and the corresponding topology accordingly.

For any  $k, i, j, \pi$ , and  $d_1^i, \dots, d_{k-1}^i$ , let  $H_{k,i,j,\pi,d_1^i, \dots, d_{k-1}^i} = F^{k-1}$  and let  $H$  be the Cartesian product of  $H_{k,i,j,\pi,d_1^i, \dots, d_{k-1}^i}$  for all possible combinations of indices, where  $j$  is an integer denoting one of  $2M_k + 1$  possible actions of player  $i$  in period  $k$ . Note that this Cartesian product is finite. For any  $h \in H$ , define its norm  $\|h\|$  as the sum of the norms of all its individual elements, and define the induced distance and topology accordingly. Note that  $H$  is a locally convex topological vector space. Note also that set  $\Xi$  of possible strategy profiles is a nonempty, compact, and convex subset of space  $H$ : it is the set of profiles  $S$  such that for any  $k, i, j, \pi$ , and  $d_1^i, \dots, d_{k-1}^i$ , element  $S_{kij}(\pi, d_1^i, \dots, d_{k-1}^i)$  is a non-negative function and that for any  $k, i, \pi$ , and  $d_1^i, \dots, d_{k-1}^i$ , the sum  $\sum_{j=1}^{2M_k+1} S_{kij}(\pi, d_1^i, \dots, d_{k-1}^i)$  is a constant function equal to 1 everywhere. Compactness follows from the facts that any infinite sequence of functions  $f^n \in F^n$  has a subsequence converging pointwise to some limit function  $f^*$ , and that under the topology induced by the norm above, pointwise convergence of a sequence of functions implies convergence in that topology.

Take any strategy profile  $S = (S_1, \dots, S_I)$ . We say that strategy  $Z_i$  of player  $i$  is a best response to  $S$  if in every continuation game, after every possible history (including the ones where

player  $i$  has played actions inconsistent with  $S$ ), for any other strategy  $Z'_i$  of player  $i$ , the expected payoff from playing  $Z_i$  is at least as high as the expected payoff from playing  $Z'_i$ , provided that prices are updated using pricing rule  $Y$  corresponding to  $S$  and other traders follow their strategies prescribed by  $S$ . Let  $BR_i(S)$  be the set of best responses of player  $i$  to strategy profile  $S$  and let  $BR(S) = BR_1 \times \cdots \times BR_I$ . We need to show that there exists strategy profile  $S$  such that  $S \in BR(S)$ .

The existence of a fixed point  $S$  of mapping  $BR$  follows from a generalization of the Kakutani fixed point theorem to general (infinite-dimensional) spaces.<sup>1</sup> To use the theorem, we need to show that  $BR$  has a closed graph and for any  $S$ , set  $BR(S)$  is convex and non-empty.

Take any player  $i$ , any period  $k$ , any history  $h$  observed by player  $i$  up to  $k$ , any strategy profile  $S$ , and any strategy  $S'_i$  of player  $i$ . Let  $W(i, k, h, S_i, S)$  denote the expected continuation profit of player  $i$ , starting with period  $k$ , following history  $h$ , when he plays according to strategy  $S_i$ , other players play according to their strategies in  $S$ , and the pricing rule is consistent with  $S$ . Note that function  $W$  is continuous in both  $S_i$  and  $S$ . This, together with compactness of set  $\Xi$ , implies that for any  $S$ ,  $BR(S)$  is non-empty and that the graph of  $BR$  is closed.

To see that for any  $S \in \Xi$ , set  $BR(S)$  is convex, let  $Z', Z'' \in BR(S)$  and let  $Z_i = \alpha Z'_i + (1 - \alpha) Z''_i$  for some  $\alpha \in (0, 1)$ . Let us show that  $Z_i$  is a best response to  $S$  after any history. The proof is by induction, going from the last period forward. Let  $k = K$ , and take any history up to  $k$ . Following that history, any action taken with positive probability under either  $Z'$  or  $Z''$  has to result in the same expected payoff, and so after any history, the expected continuation payoff from  $Z'$ ,  $Z''$ , and  $Z$  has to be the same. Suppose we have shown that after any history up to period  $k = k' + 1$ , expected continuation payoffs under  $Z'$ ,  $Z''$ , and  $Z$  are the same. Take any history  $h$  up to period  $k = k'$ . By assumption, the continuation payoffs under  $Z'$  and  $Z''$  are the same (and the best attainable). Suppose the continuation payoff under  $Z$  is less than that (it cannot be greater). Consider “strategy”  $\hat{Z}$ , under which, following history  $h$  player  $i$  plays according to  $Z'$  with probability  $\alpha$  and plays according to  $Z''$  with probability  $1 - \alpha$  (this is not a strategy according to our definition, but we can still compute expected payoffs under that behavior rule). Clearly, the expected continuation payoff from  $\hat{Z}$  is equal to that from  $Z'$  and  $Z''$ . Moreover, after any history in period  $k + 1$ , the expected continuation payoff from  $\hat{Z}$  is equal to that from  $Z'$  and  $Z''$ , and thus, by induction hypothesis, to that from  $Z$ . But in period  $k$ , the probability distribution of actions of player  $i$  under  $\hat{Z}$  and  $Z$  are identical, and thus the expected payoff in period  $k$  under the two strategies is also the same. Therefore, the continuation payoff following history  $h$  under strategy  $Z$  is the same as under  $\hat{Z}$ , which in turn is equal to that under  $Z'$  and  $Z''$ , and therefore  $Z$  is also a best response.

Thus, mapping  $BR(S)$  has a fixed point  $S^*$ . ■

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<sup>1</sup>[http://en.wikipedia.org/wiki/Kakutani\\_fixed\\_point\\_theorem#Infinite\\_dimensional\\_generalizations](http://en.wikipedia.org/wiki/Kakutani_fixed_point_theorem#Infinite_dimensional_generalizations)

### 3 Main results

Consider now the following trading game  $\Gamma$ . Take any integer  $M > 0$  and any positive real numbers  $d$  and  $\sigma$ . In every period  $k < K = \infty$ , let  $t_k = \frac{1}{2^k}$ ,  $M_k = M$ ,  $\sigma_k = \frac{\sigma}{\sqrt{2^k}}$ , and  $d_k = \frac{d}{\sqrt{2^k}}$ . Players, security, and information structure are the same as before.

#### 3.1 Existence

The next result shows that an equilibrium in this game always exists.

**Theorem A.2** *Game  $\Gamma$  has a perfect Bayesian equilibrium.*

**Proof.** Consider sequence  $\Gamma_K$  of games obtained from  $\Gamma$  by removing all trading periods following period  $K$ . Each of these games has an equilibrium,  $S_K^*$ . Consider sequence  $S_K^*$ . Take any period  $K'$ , and for  $K \geq K'$  let  $S_{K,K'}^*$  denote the sequence of strategy profiles  $S_K^*$  restricted to the first  $K'$  periods. By the compactness of the set of strategy profiles in game  $\Gamma_{K'}$ , sequence  $S_{K,K'}^*$  has a limit point for any  $K'$ . We can now construct equilibrium  $S^*$  of game  $\Gamma$  as follows. Take the limit point of  $S_{K,1}^*$ , (say,  $Z_1$ ), and take subsequence of equilibria  $Seq_1$  converging to it. In period 1, under  $S^*$ , players act according to  $Z_1$ . Now, consider the restriction of  $Seq_1$  to the first two periods. In that subsequence, strategies in the first period converge to  $Z_1$ , by construction. Strategies in the second period also have a fixed point (call it  $Z_2$ ) and a subsequence  $Seq_2$  of  $Seq_1$  converging to it. In period 2, under  $S^*$ , let players act according to  $Z_2$ . Proceeding analogously, we add  $Z_3, Z_4$ , etc. to  $S^*$ , ending up with a profile of strategies.

The fact that  $S^*$  is a perfect Bayesian equilibrium follows from the facts that (i) game  $\Gamma$  is continuous at infinity and (ii) as in the proof of Theorem 1, “counterfactual” continuation payoffs  $W(i, k, h, S_i, S)$  in any finite restriction of  $\Gamma$  are continuous in strategies. ■

#### 3.2 Information Aggregation

The last result shows that in any equilibrium of game  $\Gamma$ , if security  $X$  is separable, information about it gets aggregated.

**Theorem A.3** *Take any equilibrium  $S^*$  of game  $\Gamma$  and the corresponding stochastic process  $y_k$ . For any  $\epsilon > 0$ , there exists  $K'$  such that for any  $k > K'$  and any realization  $\omega$  of the nature’s draw, the probability that  $|y_k - X(\omega)| > \epsilon$  is less than  $\epsilon$ .*

**Proof.** The proof follows the same steps as the proof of the first statement of Theorem 1 for game  $\Gamma^K$  in the paper.

Step 0 is unchanged.

Step 1 is almost the same, except that in the last equation,  $Var(u_{k'})$  is now equal to  $\frac{V_{\sigma,d}}{2^{k'}}$  instead of being equal to  $\frac{1}{2^{k'}}$ , where  $\frac{V_{\sigma,d}}{2^{k'}}$  is the variance of the discretized normal distribution with mean 0, variance parameter  $\sigma^2$ , and unit of discretization  $d$ . This difference, however, does not change the conclusion (the last paragraph) of the step.

Steps 2-6 are unchanged.

The remaining steps change more substantially, and are thus presented below.

**Step 7.**

**Lemma A.1** *For any  $A > 0$ ,  $\mu > 0$ ,  $\sigma > 0$ , there exists  $\nu > 0$  such that for any  $z \in [-A, A]$  and any set  $S \subset \mathbb{R}$  whose probability under  $N(0, \sigma^2)$  is less than or equal to  $\nu$ , the probability of set  $(S - \sigma z)$  under  $N(0, \sigma^2)$  is less than or equal to  $\mu$ .*

**Proof.** By rescaling, it is enough to prove the lemma for  $\sigma = 1$ . Let  $\Phi(\cdot)$  denote the cdf of  $N(0, 1)$  and, without loss of generality, assume that  $\mu \leq 1$ . Let  $u$  be the solution of equation  $\Phi(u - A) = 1 - \frac{\mu}{4}$ . Let  $\nu_0 = \frac{\mu}{2} e^{-\frac{u^2}{2}}$ . For the rest of the proof, “the probability of a set” refers to its probability under  $N(0, 1)$  and, slightly abusing notation, is denoted “ $\Phi$ .”

Take any set  $S$  whose probability is less than or equal to  $\nu_0$ . Take any  $z \in [-A, A]$ . Then  $\Phi(S - z) = \Phi((S - z) \cap [-u + A, u - A]) + \Phi((S - z) \cap ((-\infty, -u + A) \cup (u - A, \infty)))$ .

By the choice of  $u$ ,  $\Phi((S - z) \cap ((-\infty, -u + A) \cup (u - A, \infty))) \leq \Phi((-\infty, -u + A) \cup (u - A, \infty)) = \frac{\mu}{2}$ .

Also,  $\Phi(S \cap [-u, u]) \leq \Phi(S) \leq \nu_0$ . The density of the normal distribution is at least  $\frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}$  everywhere on  $[-u, u]$ , and so the Lebesgue measure of  $S \cap [-u, u]$  is at most  $\frac{\nu_0}{\frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}}$ . Hence, the Lebesgue measure of  $((S - z) \cap [-u + A, u - A]) \subset ((S - z) \cap [-u - z, u - z]) = ((S \cap [-u, u]) - z)$  is also at most  $\frac{\nu_0}{\frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}}$ . But then, since the density of the normal distribution is at most  $\frac{1}{\sqrt{2\pi}}$  everywhere, the probability of  $(S - z) \cap [-u + A, u - A]$  under the normal distribution is at most

$$\frac{1}{\sqrt{2\pi}} \frac{\nu_0}{\frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}} = \frac{\nu_0}{e^{-\frac{u^2}{2}}} = \frac{\mu}{2}.$$

Therefore,  $\Phi(S - z) = \Phi((S - z) \cap [-u + A, u - A]) + \Phi((S - z) \cap ((-\infty, -u + A) \cup (u - A, \infty))) \leq \frac{\mu}{2} + \frac{\mu}{2} = \mu$ . ■

**Step 8.** Let  $\lambda = \frac{d}{2}$  (recall that  $\frac{d}{\sqrt{2}^k}$  is the smallest positive number of shares an informed trader can buy at time  $t_k$ ). Take a small  $\epsilon > 0$  and a large  $k$  such that (i) the probability that player  $i$  buys or sells more than  $\lambda/\sqrt{2}^{k+1}$  units of the security at time  $t_{k+1}$ , conditional on his information being  $\pi_a$  and  $Q_k$  being in the  $\frac{\phi}{2}$ -neighborhood of  $r$ , is less than  $\epsilon$  and (ii) in the equilibrium under consideration, the expected continuation payoff of player  $i$  after period  $t_k$  is less than  $\epsilon/\sqrt{2}^k$ . Consider the following “alternative” continuation strategy for player  $i$ : in periods  $t_{k+2}$  and later, or in period  $t_{k+1}$  when his information is not  $\pi_a$  or  $Q_k$  is not in the  $\frac{\phi}{2}$ -neighborhood of  $r$ , do not trade; otherwise, buy  $\frac{d}{\sqrt{2}^{k+1}}$  units of the security. Note that when player  $i$  buys  $\frac{d}{\sqrt{2}^{k+1}}$  units of the security under this alternative strategy, with probability close to one, under the original strategy he was buying or selling at most  $\frac{d}{2\sqrt{2}^{k+1}}$  units of the security, which means that with probability close to one, he was not buying or selling any units of the security. So, in that case, the total amount demanded by the strategic traders in period  $t_{k+1}$  changes by  $\frac{d}{\sqrt{2}^{k+1}}$ .

Let  $\tilde{u}_{k+1}$  be the draw from  $N(0, \frac{\sigma^2}{2^{k+1}})$  used to generate discretized noise trader demand  $u_{k+1}$ . Let  $U$  be the set of draws of  $\tilde{u}_{k+1}$  such that after these draws, under the original strategy of player  $i$ , market makers' belief  $Q_{k+1}$  was in the  $\phi$ -neighborhood of  $r$  with probability at least  $1 - \epsilon$  (conditional on player  $i$ 's information being  $\pi_a$  and  $Q_k$  being in the  $\frac{\phi}{2}$ -neighborhood of  $r$ ). The new strategy of player  $i$  shifts set  $U$  by  $\frac{d}{\sqrt{2^{k+1}}}$  to the left. Thus, by Lemma A.1, under the alternative strategy of player  $i$ , the probability that  $Q_{k+1}$  is in the  $\phi$ -neighborhood of  $r$  is also close to 1, for a sufficiently small  $\epsilon$ . But then the expected continuation payoff of player  $i$  from this alternative strategy is close to or greater than  $\tau\lambda\delta_1\rho/\sqrt{2^k}$ , which is greater than  $\epsilon/\sqrt{2^k}$ , contradicting the assumption that the original strategy was optimal for player  $i$  in the continuation game following every history. ■