INFORMATION AGGREGATION IN COMMON VALUE AUCTIONS

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This paper examines the process of price formation in the presence of asymmetric information. We examine auctions in which a common value good is auctioned to a large number of bidders. We provide a simple characterization of the limiting distribution of prices that enables us to examine different mechanisms under various information structures.

1 Introduction

A key question in information economics and finance is whether prices aggregate information in a competitive environment. That is, when agents are endowed with private information, does competition lead in the limit to prices which would occur if all information were public?. This paper examines the asymptotic properties of prices in common value auctions when the number of bidders becomes large.

The main idea in this paper is to separate the issue of information aggregation from the issue of revenue efficiency. We also distinguish between the amount of information prices contain, and the degree to which prices approximate the asset's value. Using this approach we get a simple characterization of the limiting distribution of prices. We are not only able to answer whether prices converge to the asset's value, but also get an explicit form for the limiting distribution. Prices converge to the expected value of the asset conditional on the information possessed by what we call the "pivotal bidder." This provides insights on how the auction format and the information structure influence the limiting behavior of prices. It shows that the limiting distribution of prices can be derived from the statistical properties of certain order statistics. This lets us abstract from the specific equilibrium bidding strategies. For example, we show that while the first and second price auction have different equilibrium strategies they share a similar limiting distribution of prices. These auctions yield different limiting distributions from auctions such as the English auction or k_n auction. Our results also imply that increasing the amount of information that is available to the pivotal bidder increases the extent to which prices aggregate information. In a sequential auction in which bidders' observe other bidders' actions, the pivotal gets more information. As a result, prices better reflect the value of the asset.

In section 4 we simplify some known results in the literature. We derive some of the results in Pesendorfer and Swinkels (1997) regarding the k_n unit auction. We also examine the first and second price auctions that were studied by Wilson (1977), and by Milgrom (1979),(1981). As most of the existing literature this section focuses on cases in which the quality of individual signals does not depend on the number of bidders. We also provide an example in which the accuracy of an individual signal decreases with the number of bidders. It suggests the some of the results regarding the k_n unit auction might depend on keeping the quality of signals independent of the number of bidders.

2 Setup and Notation

An asset is being sold via an auction. The asset is a common value good, that is, all bidders place the same value on the asset. We analyze the effects associated with having a large number of bidders who possess private information. For that reason, we consider a sequence of auctions. In the auction indexed by n, an asset of value V_n is being sold to n bidders. We assume that the value is bounded between zero and one. Unlike sequential trading models, we treat these auctions as separate games so that in each auction there is a new set of bidders to which a different asset is being sold. We assume bidders to be risk neutral, and asymmetrically informed. Each bidder is endowed with a private signal regarding the asset's value. We denote bidder $i's$ signal in the $n-th$ auction by $S_{n,i} \in [0,1]$, we denote the $l-$th order statistic of $\{S_{n,i}\}_{i=1}^{n}$ by $Y_n(l)$, and let $Y_n^{-i}(l)$ denote the $l-$th order statistic when we exclude bidder $i's$ signal, $S_{n,i}$. We denote the price in the $n-th$ auction by P_n, and the bidding strategy in the $n-th$ auction by $b_n(\cdot)$. We make the following assumptions:

- **A1**: The distribution of the asset's value, V_n, does not depend on n; that is $F_{V_n}(\cdot) = F_{V_{n'}}(\cdot) \equiv F_V(\cdot)$, where $F_{V_n}(\cdot)$ denotes the c.d.f. of V_n. We assume that both density functions $f_V(\cdot)$ and $f_s(\cdot|V_n = v)$ are well defined.

- **A2:** Bidders are identical ex-ante; that is, the distribution of signals is symmetric.

- **A3:** (MLRP) For each n, $s' > s$ and $v' > v$

$$\frac{f_n\left(S_i = s'|V_n = v'\right)}{f_n\left(S_i = s|V_n = v'\right)} > \frac{f_n\left(S_i = s'|V_n = v\right)}{f_n\left(S_i = s|V_n = v\right)}$$

Assumption A1 implies that a fixed amount of uncertainty regarding the asset's value exists. In particular, it implies that the expected value of the asset does not depend on n, that is, $E\left(V_n\right) \equiv \bar{V}$ for all n. Assumptions A2 and A3 let us refer to standard equilibria for the auctions we consider. We examine four standard auction formats:

- *A first price auction*: Bidders submit simultaneous sealed bids, the highest bidder gets the asset and pays his bid.

- *A second price auction:* Bidders submit simultaneous sealed bids, the highest bidder gets the asset and pays the second highest bid.

- *A k_n unit auction*: k_n identical units are sold. Bidders submit simultaneous sealed bids for a single unit. The k_n highest bidders get each a single unit and each pays the $(k_n + 1) - st$ highest bid.

- *An English auction:* The auctioneer continuously raises prices. Bidders observe when other bidders drop out. The last bidder to quit, gets the asset and pays the stopping price.

We focus in this paper on the standard **symmetric** equilibria described by Milgrom (1981) and Milgrom and Weber (1982). For example, in the second price auction the equilibrium strategy for an agent who has a signal s is to bid $b_n\left(s\right) = E\left(V_n|Y_n\left(2\right) = Y_n\left(1\right) = s\right)$.

3 Analysis

In all four auctions, a bidder can avoid winning the auction by submitting a sufficiently low bid. Hence, bidders do not lose on average; that is

$$E(P_n) \leq \bar{V}$$

The seller hopes, however, that the average difference between the price and the asset's value will disappear as the number of bidders increases. We call such auctions *competitive* auctions:

DEFINITION 1 *An auction is competitive if* $E(P_n) \to \bar{V}$.

One of our objectives is to examine whether prices converge to the asset's value as the number of bidders increases. For that purpose, we introduce the following definition:

DEFINITION 2 *A competitive auction aggregates information if* $P_n - V_n \to 0$ *in probability.*

We also inquire whether prices contain enough information to determine the asset's value:

DEFINITION 3 *An auction is informative if* $E(V_n|P_n) - V_n \to 0$ *in probability.*

Some relations follow almost immediately from these definitions while others depend on the particular economic setup. For example, if prices converge to the true value of the asset then the auction is *informative.* The reverse, however, is not necessarily true. For instance, if $P_n = \frac{n-1}{n}\bar{V} + \frac{1}{n}V_n$ then $P_n \to \bar{V}$ but $E(V_n|P_n) = V_n$; that is, an auction might be *informative* without *aggregating information.* One implication of our analysis is that for *competitive* auctions this is not the case.

In all auctions we consider, there exists a unique bidder whose bid is taken to be the price. We refer to this bidder as the pivotal bidder. For example, in the second price and in the English auction, the pivotal bidder is the one with the second highest signal.[2] With the exception of the first price auction, the pivotal bidder does not actually receive the asset. Still, because of the 'winner's curse' in all of these auctions the pivotal bidder quotes a price that is lower than the expected value of the asset conditioned on his information set; we denote this information set by $\mathcal{F}_n$. In the first price auction, the pivotal bidder is endowed with the first order statistics, that is, $\mathcal{F}_n = Y_n(1)$, in a second price auction, $\mathcal{F}_n = Y_n(2)$ and in a k_n unit auction, $\mathcal{F}_n = Y_n(k_n)$. In an English auction the pivotal bidder finds out in equilibrium the collection of signals that are lower than his signal[3], hence, $\mathcal{F}_n = \{Y_n(i)\}_{i=2}^{n}$. We argue that:

LEMMA 1 $P_n \leq E\left(V_n|\mathcal{F}_n\right)$

Proof. is given in the appendix ■

Using Lemma 1 we conclude that in a competitive auction the limiting distribution of prices is the same as of the expected value of the asset conditional on $\mathcal{F}_n$. That is,

THEOREM 1 *If the auction is competitive then* $P_n - E\left(V_n|\mathcal{F}_n\right) \to 0$ *in probability*

Proof. The claim follows from the fact $E\left(P_n\right) \to \bar{V}$ in competitive auctions. In these cases since $E\left(E\left(V_n|\mathcal{F}_n\right)\right) = E\left(V_n\right) = \bar{V}$ and $P_n \leq E\left(V_n|\mathcal{F}_n\right)$, we conclude that $P_n - E\left(V_n|\mathcal{F}_n\right) \to 0$ in probability. ■

In all these auctions the price is a function of the pivotal bidder's information set. Hence, a direct implication of Theorem 1 is that in a competitive auction the limiting properties of P_n are the same as of $E\left(V_n|P_n\right)$:

THEOREM 2 *If the auction is competitive then* $E\left(V_n|P_n\right) - P_n \to 0$ *in probability.*

Proof. From Theorem 1 we know that $P_n - E\left(V_n|\mathcal{F}_n\right) \to 0$. Conditioning on P_n we conclude that: $E(P_n|P_n) - E\left(E\left(V_n|\mathcal{F}_n\right)|P_n\right) \to 0$. Since P_n is a function of $\mathcal{F}_n$ we conclude that $E\left(E\left(V_n|\mathcal{F}_n\right)|P_n\right) = E\left(V_n|P_n\right)$, and, the claim then follows from the fact that $E(P_n|P_n) = P_n$ ■

This result implies a strong relation between how informative is an auction and the extent to which price approximate the asset's value. It implies that **if the auction is competitive, it aggregates information if and only if it is informative**. Another key insight is that in a competitive auction the limiting distribution of prices is determined by the information available to the pivotal bidder. Hence, we can derive the limiting distribution of prices without relying on the exact functional form of the bidding strategy. The computation of the limiting distribution of prices is a statistical question of computing conditional expectations. Finally, we note that by increasing the amount of information that the pivotal bidder has, the auction that better aggregates information. Specifically, since the asset's value is bounded we conclude that:

COROLLARY 1 *Assuming that both auctions are competitive, if the second price or any* k_n*-unit auction aggregates information then the English auction also aggregates information.*

4 Application to the Four Auction Formats

In this section we examine the four auction formats under a more concrete setup. We use a standard setup (see. Pesendorfer and Swinkels (1997))

- **A4**: For each n, signals are i.i.d. conditional on the asset's value. The conditional density of signals does not depend on n; we denote it by $f(s|v)$.
- **A5**: There exists some $\gamma > 0$ so that $\frac{f(s|v)}{f(s'|v)} < \gamma$ for any s, s', and v

Assumption A4 implies that the quality of a single signal does not vary with the number of agents. While this assumption may or may not hold,[4] it does simplify the analysis. It provides a simple way to describe the distribution of signals as we add more bidders. Assumption A5 implies an upper bound on the amount of information contained in a single signal. We first argue that:

THEOREM 3 *Under Assumptions A1-5, , all four auctions are competitive.*

Proof. is given in the appendix ■

The above result together with Theorems 1 and 2 enables us to characterize the limiting distribution of prices in these auctions without relying on the specific auction format and equilibrium.

- *First price auction*: We argue that if Assumption A5 holds, then this auction does not aggregate information. To see this we have to show that $E(V_n|Y_n(1)) - V_n \nrightarrow 0$. It follows from the fact that $E(V_n|Y_n(1))$ is bounded away from one:

  $$E(V_n|Y_n(1)) \leq E(V_n|Y_n(1)=1) = E(V_n|S_i=1) < 1$$

  We also note that in this case $P_n \nrightarrow \bar{V}$. Otherwise, with a large number of bidders an agent who has a high signal would deviate by offering a price that is strictly higher than $\bar{V}$.

- *Second price auction*: While the equilibrium strategy in this auction is quite different from that of a first price auction, the limiting distribution of prices is very similar. The reason is that the limiting distribution of $E(V_n|Y_n(2))$ is similar to that of $E(V_n|Y_n(1))$.

- $\underline{k_n\ \textit{unit auction}}$: Consider the case of $k_n = \lfloor \frac{n}{a} \rfloor$ where a is some fixed constant[5] The weak law of large numbers implies that

$$Y_n \left(\lfloor \frac{n}{a} \rfloor + 1 \right) \to F_{s|v}^{-1} \left(1 - \frac{1}{a} \right) \text{ in probability}$$

  That is, the $\lfloor \frac{n}{a} \rfloor + 1$ highest signal converges to a number where the cumulative distribution function equals $1 - \frac{1}{a}$. The MLRP condition (Assumption A3) implies that $F_{s|V}^{-1} \left(1 - \frac{1}{a} | V = v \right)$ is strictly increasing in v. We conclude that

$$E \left(V_n | Y_n \left(k_n + 1 \right) \right) - V_n \to 0$$

  which implies that these auction aggregate information.

- *English auction*: Using Corollary 1, we conclude that the English auction also aggregates information.

The above results demonstrate the difference between a static auction in which bidders submit sealed bids (e.g., first price auction, second price auction) and a dynamic auction in which bidders submit sequential bids (e.g., English auction). In a static auction the amount of information the pivotal bidder has is limited to his signal and the fact that he is pivotal. This is why the first and second price auctions fail to aggregate information. It is interesting to note that by Theorem 2 since $P_n \nrightarrow \bar{V}$ in a first price auction, we conclude that $E(V_n|Y_n(1)) \nrightarrow \bar{V}$. This is despite the fact that $Y_n(1)$ converges to one regardless of the asset's value.

The example of a k_n unit auction demonstrates that in some cases the information of being pivotal is sufficient to determine the asset's value. However, as we shall see in the next section, it strongly depends on the assumption that the quality of information does not depend on the number of bidders.

5 Example-Decreasing Accuracy of Signals

Under Assumptions A4 and A5 we have just shown that if $k_n = \lfloor \frac{n}{2} \rfloor$ then a k_n unit auction aggregates information. The argument is based on the weak law of large numbers that assumes that the variance of signals does not depend on n. Our goal here is to describe an example in which the accuracy of signals decreases with an increase in the number of bidders. While the example is somewhat extreme and artificial it shows that the result regarding the k_n unit auction might critically depend on keeping the accuracy of signals fixed. Consider a case in which signals are uniformly distributed on $[0,1]$ and the asset's value is given by:

$$V_n\left(\{S_n\}_{i=1}^n\right) = \begin{Bmatrix} 1 \text{ if } \frac{1}{n}\sum_{i=1}^n S_{n,i} > \frac{1}{2} \\ 0 \text{ otherwise} \end{Bmatrix} \tag{1}$$

The distribution of V_n does not depend on the number of bidders, n and Assumptions A1-3 hold. However, the accuracy of signals decreases as a fixed amount of information is being divided among an increasing number of bidders. The information conveyed in the median being x is that $n/2$ signals are distributed uniformly on $[0,x]$ and $n/2$ signal are distributed uniformly over $[x,1]$. Using the central limit theorem one can show that the median signal converges to $\frac{1}{2}$ at a rate of $n^{-0.5}$. That is, there is a positive probability that the median signal is in $\left[\frac{1}{2}-n^{-0.5},\frac{1}{2}+n^{-0.5}\right]$. Conditional on the median being in this interval, the central limit theorem implies that there is uncertainty regarding V_n. Hence, The median signal (or any other quantile) does not provide enough information to determine the asset's value. We conclude that the k_n fails to aggregate information in this case.

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6 Appendix

Proof of Lemma 1:

- In a first price auction, the price is the winning bid. In this auction a bidder chooses his bid so as to maximize his expected profit conditional on him winning the auction. To extract non-negative profits from the auction, the bid should be below the expected value of the asset conditioned on him having the highest signal. That is, $b_n(s) \leq E(V_n|Y_n(1)=s)$, this implies that $P_n = b_n(Y(1)) \leq E(V_n|Y_n(1))$.
- In a second price auction, the price is the bid of the bidder with the second highest signal. Hence, the price is the expected value of the

asset conditional on the second highest signal $Y_n(2)$ and on the highest signal being equal to the second highest signal. That is,

$$P_n = E\left(V_n | Y_n(2), Y_n(1) = Y_n(2)\right)$$

since $Y_n(1) \geq Y_n(2)$ we conclude that $P_n \leq E\left(V_n | Y_n(2)\right).$

- The proof for k_n unit auction is similar to the proof for a second price auction. The only difference is that we replace $Y_n(1)$ by $Y_n(k)$ and $Y_n(2)$ by $Y_n(k+1)$.
- The proof for an English auction is again very similar to the proof for a second price auction. The only difference is that the pivotal bidder gets the information $\{Y_n(i)\}_{i=n}^{3}$ in addition to his signal, $Y_n(2)$.

■

Proof of Theorem 3:

Consider first the first price auction. Let $b_n^*(s)$ denote the value of the asset conditioned on the highest signal being s, that is $b_n^*(s) \equiv E\left(V_n | Y_n(1) = s\right).$ Then the expected difference between value and price is given by:

$$\bar{V} - E(P_n) = \int_0^1 h_n(s)\, dF_{Y_n(1)}(s)$$

where $h_n(s) = b_n^*(s) - b_n(s)$.

Step 1: $h_n(1) \to 0$

First note that $b_n^*(1) = E(V_n | s_i = 1)$ which implies that $b_n^*(1)$ does not depend on n. Assume by contradiction that there exists some $\delta > 0$ so that $b_n(1) < b_n^*(1) - \delta$ for large n. This implies that there exists some $s' < 1$ so that $b_n(1) < b_n^*(s')$ and hence bidder i can bid $b_n^*(s')$ whenever his signal $s_i \geq s'$. This would guarantee him a positive profit that is bounded away from zero. However, note that since the difference between price and value is bounded by one, the bidder's profit by following the equilibrium strategy is bounded by $\frac{1}{n}$. Hence, this deviation is profitable for large n, which leads to a contradiction.

Step 2: For any s we have $h_n(1) \geq pr\left(Y_n^{-i}(1) < s | s_i = 1\right) \cdot h_n(s)$

Whenever bidder i gets a signal $s_i = 1$ and follows the equilibrium strategy he gets a payoff of $h_n(1)$. If instead he submits a lower bid, $b(s)$ his profit is given by

$$pr\left(Y_n^{-i}(1) < s | s_i = 1\right) \cdot h_n(s)$$

Step 3: $\underline{F_{Y_n(1)}(s) < \gamma \cdot pr\left(Y_n^{-i}(1) < s|s_i = 1\right)}$

Note that:

$$pr\left(Y_n^{-i}(1) < s\right) = \int f_v(v) F(s|v)^{n-1} dv$$

and that

$$pr\left(Y_n^{-i}(1) < s|s_i = 1\right) = \int f(v|s_i = 1) F(s|v)^{n-1} dv$$

The proof follows then from the fact that $pr(Y_n(1) < s) < pr\left(Y_n^{-i}(1) < s\right)$ and Assumption A5.

Step 4: $\underline{\bar{V} - E(P_n) \to 0}$

Let us fix an $\varepsilon > 0$ and pick n large enough so that so that $h_n(1) < \varepsilon^2$. It can be done using step 1. Let s_n' be defined by $pr\left(Y_n^{-i}(1) < s_n'|s_i = 1\right) = \varepsilon$. From step 2 we know that for $s > s_n'$ we have $h_n(s) < \varepsilon$. We also know that $h_n(s) < 1$ for any s, hence :

$$\begin{aligned} \bar{V} - E(P_n) &= \int_0^{s_n'} h_n(s)\, dF_{Y_n(1)}(s) + \int_{s_n'}^1 h_n(s)\, dF_{Y_n(1)}(s) \\ &\leq \gamma \cdot \varepsilon + \varepsilon = (\gamma + 1)\varepsilon \end{aligned}$$

As ε can be chosen arbitrarily small, the theorem is proved.

Using the 'Linkage principle' we conclude that the second price and English auctions are also competitive. This however, does not cover the case for a general k_n unit auction. We first show this for a case in which $\frac{k_n}{n} \to a$. Since one can decompose the sequence of $\{k_n\}$ to subsequences, the case of $\frac{k_n}{n} \to a$ would imply that it holds also for the general case. Similar to the proof for the first price auction we define $b_n^*(s) \equiv E(V_n|Y_n(k_n + 1) = s)$, $h_n(s) = b_n^*(s) - b_n(s)$. We also define $b_n^{**}(s) \equiv E(V_n|Y_n(k_n) = s)$. Note that our MLRP assumption implies that:

$$b_n^{**}(s) < b_n(s) < b_n^*(s)$$

Also note that $b_n^{**}(s), b_n^*(s) \to v^*$ where v^* satisfies $F(s|v^*) = 1 - \frac{1}{a}$. Hence, we conclude that $h_n(s) \to 0$. The proof then follows from the fact that

$$\bar{V} - E(P_n) = \int_0^1 h_n(s)\, f_{Y_n(k_n)}(s)\, ds$$

and the fact that since the limiting distribution of $Y_n(k_n)$ is given by $F^{-1}(1 - 1/a|v)$ we conclude from Assumption A1 that $f_{Y_n(k_n)}(s)$ converges to a finite number. ■

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Notes

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[2] Note that the equilibrium in the first price auction, described in the previous section, is in monotone increasing strategies. This implies that the bidder who has the highest signal also submits the highest bid.

[3] This follows since, in equilibrium, the price at which each bidder quits is monotone increasing in his signal, see Milgrom and Weber (1982).

[4] One can imagine that if information is costly, there is less incentive to acquire information as the number of agents increases. As a result, it is possible that the quality of signals deteriorate.

[5] $\lfloor \frac{n}{a} \rfloor$ denotes the biggest integer that is still smaller than n/a.