

On the Informational Inefficiency of Discriminatory Price Auctions *

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Abstract

We analyze bidding behavior in large discriminatory-price auctions in a common value setting where the number of objects is a non-trivial proportion of the number of bidders. We show that the average price paid in the auction is biased downward from the expected value of the objects, even in the competitive limit. We provide a full characterization of bids in the limit. We show that conditional on a signal that falls below a threshold, a bidder bids the expected value of an object conditional on the signal and winning; while conditional on any signal that lies above the threshold the bid is lower than the expected value conditional on the signal and winning.

Keywords: Auction, Competition, Discriminatory Auction, Asymptotic Efficiency, Efficient Markets, Efficient Market Hypothesis.

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1 Introduction

Efficient Market Hypotheses posit that markets aggregate the diverse information held by the agents in an economy so that prices come to reflect the expectation of the value of assets based on this information. This informational role of prices is important since it aids in efficient investment decisions. As such, it is important that we understand the conditions under which efficient market hypotheses hold.

In this paper we examine price behavior in discriminatory auctions with many agents and many goods, with a focus on the relationship between equilibrium prices and the true value of the goods. There are at least two reasons for being interested in auctions when examining the issues mentioned above. The first is that auctions are a widely used method of selling (and buying) goods and services. For instance, the discriminatory price auction format is used to sell treasury securities in many countries. The second is that the strategic reasoning that underlies the analysis of auctions is present even in markets that are not formally organized as auctions. While the previous literature on information aggregation in large auctions focuses on the uniform price auction; their widespread use suggests that we should also examine discriminatory price auctions.

We examine common-value discriminatory auctions of many units of a good to many bidders who each demand a single unit. We are interested in the competitive limit and hence analyze sequences of auctions with increasing numbers of bidders. We find that bidders who are uncertain as to whether they will obtain an object compete sufficiently so that their conditional expected utility goes to zero. That is, the bids of agents whose signals are below a certain threshold converge to the expected value of the object conditional on their respective signals and what they infer from winning in equilibrium. In contrast, agents whose signals are sufficiently high so that they are certain to get an object (in equilibrium) end up with an expected utility that remains bounded away from zero even in the limit. For signals above this threshold, the bidding strategy “flattens” since agents are (nearly) certain that they will win an object. This leads to a distinct price pattern as prices differ from the pooled information expectation of the value of the object in a particular way. For example, when we integrate over all realizations, the average price is biased downward from the objects’ true value.

Before presenting our results, let us discuss the contribution of our work relative to other papers that examine information aggregation in common value auctions. Wilson (1977) and Milgrom (1979,1981) provided important results characterizing situations where equilibrium prices reflect the expected value of an asset. They examined common-value auctions with a fixed number of units of the asset for sale and a growing number of bidders, and identified conditions under which prices converge to the expected value of the asset conditional on the information of all bidders. Their results might be interpreted as “bad news” for efficient market hypotheses in the sense that the necessary and sufficient condition on the structure of information for such aggregation is a very strong one and unlikely to be met in most settings of interest.

In an important recent paper, Pesendorfer and Swinkels (1997) (PS hereafter) showed that the news is much better in settings where the number of units of the asset for sale grows when the number of bidders grows. Under quite weak assumptions on the information structure, they showed that the limiting price of a uniform price auction converges to the

expected value of the asset as if the economy had the information of all bidders. As many markets of interest, such as Treasury auctions, have both large quantities of the asset for sale and large numbers of bidders, this is “good news” for market efficiency. Apart from the auction format, the model we presents is similar to PS.

As the results of Wilson (1977) and Milgrom (1979, 1981) hold for a wide variety of auction formats,¹ one might wonder whether the PS results do as well. We demonstrate that, unfortunately, the price-efficiency result of Pesendorfer and Swinkels (1997) depends on the auction format. Specifically, we show that information aggregation fails in the context of one of the most prominent auction formats: the discriminatory price auction. We do this showing that the average winning bids in a discriminatory auction are biased downward relative to the true value of an object.

2 Model

We examine a framework that is similar to PS, except for the auction format.

We examine a sequence of discriminatory auctions with an increasing number of bidders. The sequence is indexed by n , the number of agents in the economy. $0 < k_n \leq n$ identical indivisible objects are to be sold in economy n . The highest k_n bidders are each awarded a single object and each pays his or her bid, and the other bidders pay nothing. Ties are broken by uniform randomization among the tied bidders.

We make the following assumptions.

(A1) The supply of goods is proportional to the number of bidders: $\frac{k_n}{n} \in (0, 1)$ for all n and $\frac{k_n}{n} \rightarrow \gamma \in (0, 1)$.

Each agent values a single unit at the same (common) value v , and can only buy one object.² Agents are risk neutral and agent i receives a payoff of $v - b_i$ if he or she receives an object at a bid of b_i and the value turns out to be v . Otherwise, the bidder’s payoff is 0.

The common value is described by a random variable, V , that takes on values in $[0, 1]$ and is unknown to bidders. The distribution of V is described by a density function f .

(A2) f is continuous and positive on $[0, 1]$.

Before bidding, each bidder $i \in \{1, \dots, n\}$ in economy n observes the realization of a random variable S_i that takes on values in $[0, 1]$. The S_i ’s are independently and identically distributed conditional on V . The conditional distribution of S_i given V is described by a conditional density function $g_{S_i|V}(s|v)$.

A standard affiliation condition ensures the existence of increasing and symmetric equilibria in both the discriminatory and uniform price auctions. We follow PS in assuming such an affiliation condition in addition to some continuity and differentiability conditions:

¹See Bali and Jackson (2002) and Jackson and Kremer (2003a,b) for results on the equivalence of auction formats with fixed numbers of objects for sale and large numbers of bidders.

²The analysis here extends in an obvious way to situations where agents value multiple units at the common value, even if the number they demand is stochastic; provided the distribution of quantities is symmetric across agents and independent of their signals. If the quantity that agents demand is related to their signals, however, then we do not have an expression for limiting equilibrium behavior and are not sure whether or under what circumstances the results extend.

(A3) $g_{S_i|V}(s|v)$ is continuous in (s, v) and continuously differentiable in v for all s and v .

The continuity of the conditional density function limits the potential information advantage that is obtained by any given signal. In particular, nearby signals will have nearly the same posteriors. Differentiability is assumed for technical convenience.

(A4) For every signal, s there exists a value v , so that $g_{S_i|V}(s|v) > 0$.

(A4), together with (A3), ensures that each signal has a positive probability of being observed.

(A5) $g_{S_i|V}(s|v)$ satisfies the Strict Monotone Likelihood Ratio Property (henceforth, SMLRP):

$$\frac{\partial g_{S_i|V}(s|v)}{\partial v g_{S_i|V}(s'|v)} > 0,$$

for all $s_i > s'_i$ in $[0,1]$ and all $v \in [0, 1]$, and this partial derivative is continuous.

Note that we change the standard weak inequality in the MLRP condition to be strict and we add continuity. The continuity is for technical convenience. However, the strictness in the SMLRP condition is more critical to our proof. As a simple illustration of why, consider a situation where signals are completely independent of the value of the object. That would satisfy the usual weak MLRP condition, but not SMLRP. In such a situation there exists an equilibrium where all bidders bid the expected value of the object (which is the same for all bidders), and the resulting price completely reflects the bidders' information. That would contradict our results. In order for our results to hold, it must be that there is a strict relation between signals and the realized value; which one of the implications of the strictness in SMLRP in our setting. While the SMLRP is more than what is needed (see the example below), it is sufficient for a clean proof.

Let $G_{S_i}(s)$ denote the unconditional distribution of S_i and g_{S_i} be the corresponding density. Let Y^n denote the k_n -th order statistic of the signals and Y_{-i}^n denote the k_n -th order statistic of the signals excluding i 's signal. Let $G_{V|S_i}(v|s)$ denote the distribution of V conditional on observing a signal S_i .

Define $s^*(v)$ to be the s such that

$$G_{S_i|V}(s|v) = 1 - \gamma.$$

As the equilibrium bidding strategy turns out to be increasing, this is the asymptotic cutoff signal for getting an object, conditional on $V = v$. Under our assumptions, $s^*(v)$ is well-defined and increasing in v , and $s^*(1) < 1$.

Given a realization $S_i = s$, let $v(s)$ denote the realization of V for which s is just on the margin of obtaining an object in the limit (under a sequence of increasing and symmetric equilibria). Since $s^*(v(s)) = s$ for $s \in [s^*(0), s^*(1)]$, it follows that $v(s)$ is the inverse function of $s^*(v)$. For $s \geq s^*(1)$, we follow the convention of setting $v(s) = 1$; and for $s \leq s^*(0)$ we follow the convention of setting $v(s) = 0$.

(A6) $G_{V|S_i}(v(s)|s) > 0$ for all s such that $v(s) > 0$.

(A6) implies that conditional on any signal s that has a positive cutoff value, a bidder places some probability on V lying below $v(s)$. This means that conditional on a signal with a positive cutoff value a bidder expects to win an object with some positive probability in a symmetric equilibrium in increasing strategies. While this is not a condition directly on the primitives f and $g_{S_i|V}$, it is satisfied by many information structures. For instance, it holds if $g_{S_i|V}$ has full support for each value of V .

3 The Main Result

In this setting there exists a symmetric pure-strategy Bayesian equilibrium for every n . A direct extension of the results of Milgrom and Weber (1982) for first price auctions implies that there exists a symmetric bidding strategy that is increasing in the signal. We denote this symmetric increasing equilibrium bidding strategy by $b_n(s)$ (see the appendix for an explicit characterization). We examine the limiting properties of this bidding strategy as n becomes large. Let

$$h(s) = E(V|S_i = s, v(s) \geq V).$$

Thus, we can think of $h(s)$ as the limiting expected value of an object conditional on the event that a bidder wins the auction with a signal s under the symmetric increasing bidding strategy. Our main result is:

Theorem 1 *If A1 through A6 hold, then*

$$b_n(s) \rightarrow \min \{h(s), h(s^*(1))\}.$$

The theorem characterizes the limit of the symmetric equilibrium bidding strategy. Bids for relatively low signals converge to be the expected value conditional on winning, while bids for relatively high signals converge to a cutoff price. The cutoff price is the bid that corresponds to a signal such that any signal above this level is sure to get an item in the limit. The second part of this characterization is straightforward, while the first part of the theorem - that bids for low signals converge to expected value - is more difficult to prove. We provide the proof in the appendix, and illustrate the basic ideas in an example below.

The theorem leads to the following corollary.

Corollary 1 *If A1 through A6 hold, then the expected price in the auction converges to a level that is below the expected value of the asset.*

Let us offer some observations about the price patterns that accompany the above results.

- (a) The average price would not converge to the value of the object even if all agents bid according to $h(s)$. To see this consider a case where both V and the distribution of S conditional on any $V = v$ have full supports. If all agents bid according to $h(s)$, the average winning bid could end up being above or below the actual value. For instance, if V is close to zero then most bidders will bid significantly above zero and the average price will exceed the asset's value. Conversely, when V is close to one the average price will be well below one and hence too low.

- (b) The discriminatory auction leads to a lower expected price than the uniform price auction. This follows since Pesendorfer and Swinkels (1997) have shown that the price of the uniform price auction converges to the expected value of the objects in the limit.³
- (c) In the limit, the increasing nature of the equilibrium means that it is “revealing” in the sense that someone who knows the structure of the bidding functions and sees the average price can correctly forecast the value of the asset. However, this is a weak form of “revelation”. Most importantly, such a weak form of revelation does not capture the spirit of efficient market hypotheses and does not identify mechanisms that aggregate information. The idea is that the market should come to mold prices in ways so that they reflect the value of assets; not that an observer through the daunting task of observing the appropriate bids and inverting them to deduce signals can unravel the information present in the economy.
- (d) As γ approaches 0, $s^*(1)$ approaches one and an increasing portion of the bidders end up bidding their expected value conditional on their signal. In the limit, the expected surplus that goes to bidders will go to 0. However, this does not imply that the price converges to the value. This is as we argued in (a).

To better understand our results, consider the following example.

Example 1

- The number of units for sale is $k_n = n/2$.
- The value of the good v is distributed uniformly on $[0, 1]$.
- Conditional on v , signals are distributed uniformly on $[0, v]$.⁴

In the uniform price auction (as PS have shown) prices converge to the true value of the asset V . In this example, it is easy to see that the bids in the symmetric equilibrium of the uniform auction converge to being twice the signal ($2S_i$): the median signal sets the price, and this median converges in probability to $S_i = v/2$. Thus, it follows fairly easily that the price converges to the value.

An extension of the results of Milgrom and Weber (1982) (see the appendix for more detail) leads to a characterization of a symmetric equilibrium in the discriminatory price auction described by

$$b'(s) G(Y_{-i}^n = s | S_i = s) = g(Y_{-i}^n = s | S_i = s) (v_n(s, s) - b(s)). \quad (1)$$

where $v_n(t, s) \equiv E(V | Y_{-i}^n = t, S_i = s)$. The solution to the above differential equation combined with the boundary condition $b_n(0) = 0$ yields a pure strategy, increasing, symmetric equilibrium. The bidding strategies from (1) are generally hard to compute for a

³The same holds in a wide variety of private value settings too, as shown in Jackson and Kremer (2003a).

⁴This violates (A3) as $g_{S_i|V}$ is not continuous (at $s = v$), and also violates (A5) as it only satisfies a weak inequality and is discontinuous at $s = v$. However, the example is easy to solve and thus makes the intuition transparent. The example can be perturbed slightly to satisfy our assumptions. Note that the example also illustrates that our conclusions extend beyond the assumptions we use.

finite n , as $v_n(s, s)$ involves a non-trivial computation. However, in the limit this expression simplifies tremendously. As signals are distributed uniformly on $[0, v]$, conditioned on $S_i = Y_{-i}^n(k_n) = s$ one can say with near certainty that $v = 2s$. Moreover, in the limit $G(Y_{-i}^n(k_n) = s | S_i = s) \rightarrow \frac{s}{1-s}$, while $g(Y_{-i}^n(k_n) = s | S_i = s) \rightarrow \frac{2}{1-s}$. Hence, (1) simplifies to

$$b'(s) \frac{s}{1-s} = \begin{cases} \frac{2}{1-s}(2s - b(s)) & \text{if } s < .5 \\ 0 & \text{if } s \geq .5. \end{cases}$$

The fact that the derivative $b'(s) = 0$ for $s > 0.5$ follows from the fact that in the limit the median signal is lower than 0.5 with certainty, and so raising one's bid for a signal above 0.5 only increases the payment. This is key to our analysis. All signals which give a bidder (nearly) a probability of one of getting an object must lead to (nearly) the same bid, as raising the bid above this level is pointless. The other part of the proof is that bidders who face a lower probability of getting an object are in keen competition as there are many bidders with similar information, and so they end up bidding away their surplus, which we see in the following bidding function. Using the boundary condition of $b(0) = 0$, we get the following characterization for the limiting bid function.

$$b(s) = \begin{cases} \frac{4s}{3} & \text{if } s < .5, \text{ and} \\ \frac{2}{3} & \text{if } s \geq .5. \end{cases}$$

Consider two cases,

- $v \leq 0.5$ - In this case winning bidders will have signals uniformly distributed on $[v/2, v]$ and the average signal is then $\frac{3v}{4}$. Note that since in this range the bidding strategy is linear, the average price converges to $\frac{3}{4} * \frac{4}{3}v = v$. Hence, in this example the average price converges to the asset's value. This, however, is not robust. It is partly an artifact of the structure of this particular example where conditional on $V = 0$ the distribution of signals is degenerate. As we discussed in the remarks following the theorem, it is possible for the price to exceed value at low values.
- $v > 0.5$ - In this case winning bidders are uniform on $[v/2, v]$ with the average signal is $\frac{3v}{4}$. However, in this range the bidding strategy is concave rather than linear. Hence, it follows that the average price converges to a number strictly below v .

We remark that on average the revenue per unit converges to .43 in the discriminatory auction, compared to 0.5 in the uniform auction.

4 Appendix

The following lemma is useful in the proof of Theorem 1.

Lemma 1 *Under (A1) through (A6)*

- (i) $g_{Y_n|S_i}(s|s)$ is uniformly bounded above across n and s , and

(ii) Given any $a > s^*(0)$, $G_{Y_n|S_i}(s|s)$ is uniformly bounded below across n and $s > a$.

Proof of Lemma 1:

(i) Note that $g_{Y_n|S_i}(s|s) = \int g_{Y_n|v}(s|v)g(v|s)dv$.

Let us argue that $g_{V|S_i}(v|s)$ is bounded above. $g(v|s) = \frac{g(s|v)f(v)}{\int g(s|v)f(v)dv}$. Given that $g(s|v) > 0$ for some v by (A4) and this is continuous by (A3), the denominator is positive for all s . Moreover, as it is continuous in s by (A2) and (A3), and positive everywhere on a compact set of s , it is uniformly bounded below in s . Given the continuity of the numerator in s and v , it is bounded above uniformly on a compact set. The claim that $g_{V|S_i}(v|s)$ is bounded above then follows.

So, it is enough to show that we can find M so that $\int g_{Y_n|v}(s|v)dv < M$. Note that $g_{Y_{k_n}|V}(s|v) = \frac{n!}{(k-1)!(n-k)!}G_{S_i|V}(s|v)^{k_n-1} (1 - G_{S_i|V}(s|v))^{n-k_n} g_{S_i|V}(s|v)$, since $g_{S_i|V}(s|v)$ is bounded from above it is enough to consider:

$$\int \frac{n!}{(k-1)!(n-k)!}G_{S_i|V}(s|v)^{k_n-1} (1 - G_{S_i|V}(s|v))^{n-k_n} dv$$

We change variables by letting $X(v) = G_{S_i|V}(s|v)$ and get

$$\int \frac{n!}{(k-1)!(n-k)!}X^{k_n-1} (1 - X)^{n-k_n} dX(v).$$

Let us argue that $X'(v)$ is bounded above and below. Note that under this change of variables, $\frac{\partial G_{S_i|V}(s|v)}{\partial v}$ is nonpositive under (A5), and so is bounded above by 0. It is bounded below as follows.

$$G(s|v) - G(s|v') = \int_0^s [g(s'|v) - g(s'|v')] ds'.$$

Given the continuity of $\frac{\partial g(s'|v)}{\partial v}$ on a compact set, we can find A such that

$$g(s'|v) - g(s'|v') < A(v - v'),$$

and the result then follows.

So, it is enough to consider

$$\int \frac{n!}{(k-1)!(n-k)!}X^{k_n-1} (1 - X)^{n-k_n} dx$$

Note that the integrand above is the density of the k_n order statistic evaluated at X , where the sample is drawn from the uniform distribution. Hence the integral is one.

(ii) Fix $a > s^*(0)$. By the continuity of $G_{V|S_i}$ (from (A2) and (A3)), it follows from (A6) that there exists $\delta > 0$ such that $G_{V|S_i}(v(s)|s) > \delta$ for any $s \geq a$. Since $g_{V|S_i}$ is uniformly bounded above (based on an argument similar to that behind (i)), we can find a $\delta_1 > 0$ such that $G_{V|S_i}(v(s) - \delta_1|s) > \delta_1$ for all $s \geq a$.

Next, let us argue that the SMLRP condition (A5) implies that for any $v \leq v(s) - \delta_1$ there exists $\delta_2 > 0$ so that $G(s|v) > 1 - \gamma + \delta_2$. Let $v' = v(s) - \delta_1$. It is enough to bound $\frac{1-G(s|v')}{G(s|v')}$ away from $\frac{1-G(s|v(s))}{G(s|v(s))} = \frac{\gamma}{1-\gamma}$. Note that $\frac{1-G(s|v')}{G(s|v')} = \frac{\int_s^1 g(t|v')dt}{\int_s^1 g(t|v')dt}$, $\frac{1-G(s|v(s))}{G(s|v(s))} = \frac{\int_s^1 g(t|v(s))dt}{\int_0^s g(t|v(s))dt}$

Fix an ε and note that A5 implies that $\frac{g(t_1|v')}{g(t_2|v')} > \frac{g(t_1|v(s))}{g(t_2|v(s))} (1 + \tau)$ for some $\tau > 0$ for any t_1, t_2 so that $t_1 \geq t_2 + \varepsilon$ (where we are using the continuity of the partial derivative on a compact set). This implies that

$$\frac{\int_{s+\varepsilon}^1 g(t|v') dt}{\int_0^s g(t|v') dt} > (1 + \tau) \frac{\int_{s+\varepsilon}^1 g(t|v(s)) dt}{\int_0^s g(t|v(s)) dt}$$

The reason why the above is true is that if $\frac{A_i}{B_j} > \frac{a_i}{b_j} (1 + \tau)$ for any i, j , then $\frac{\sum_j A_i}{\sum_j B_j} > (1 + \tau) \frac{\sum_j a_i}{\sum_j b_j}$, which implies that $\frac{\sum_i A_i}{\sum_j B_j} > (1 + \tau) \frac{\sum_i a_i}{\sum_j b_j}$. Also note that (A5) implies

$$\frac{\int_s^{s+\varepsilon} g(t|v') dt}{\int_0^s g(t|v') dt} > \frac{\int_s^{s+\varepsilon} g(t|v(s)) dt}{\int_0^s g(t|v(s)) dt}.$$

The claim then follows, as we can pick ε to be small, and this combined with $g(s|v)$ being uniformly bounded implies that we can make $\int_s^{s+\varepsilon} g(t|v(s)) dt$ small.

Using Chebychev's bound we conclude that conditional on $v \leq v(s) - \delta_1$, the probability that $s \geq Y_n$ is bounded away from zero across n and $s \geq a$. To show this, we argue that $Pr(Y_n \leq s|V = v)$ is bounded away from zero across s and v such that $G(s|v) > 1 - \gamma + \delta_2$. Consider n i.i.d Bernoulli random variables $\{X_i\}_{i=1}^n$ where $Pr(X_i = 1) = p \equiv 1 - \gamma + \delta_2$. Let $X = \sum X_i$. Then

$$Pr(Y_n \leq s|V = v) \geq Pr(X \geq n(1 - \gamma))$$

Note that $EX = (1 - \gamma + \delta_2)n$ and $var(X) = np(1 - p)$. We note also that $Pr(X < (1 - \gamma)n) = Pr(X < EX - \delta_2 n)$. Chebyshev's bound then implies that

$$Pr(X < EX - \delta_2 n) \leq \frac{var(X)}{(\delta_2 n)^2} = \frac{1}{n} \frac{(1 - \gamma + \delta_2)(\gamma - \delta_2)}{(\delta_2)^2},$$

which is bounded away from 1, which means that $Pr(X \geq n(1 - \gamma))$ is bounded away from 0. Thus, so is $Pr(Y_n \leq s|V = v)$. Part (ii) now follows since $G_{V|S_i}(v(s) - \delta_1|s) > \delta_1$. ■

Proof of Theorem 1: A direct extension of the results of Milgrom and Weber (1982) for first price auctions leads to the derivation of a symmetric equilibrium for the discriminatory auction as follows. If a type s pretends to have a slightly higher signal, then the additional cost is $b'_n(s)$ times the probability that s was among the highest k_n signals. The expected benefit from such a deviation is winning an object in the marginal event where s was tied with the k_n -th signal. The probability of this is the density of the k_n -th highest signal at s conditional on observing $s_i = s$. The benefit conditional on this event is the difference between value and bid. This leads to the following differential equation.

$$b'(s) G(Y_{-i}^n = s|S_i = s) = g(Y_{-i}^n = s|S_i = s) (v_n(s, s) - b(s))$$

where $v_n(t, s) \equiv E(V|Y_{-i}^n = t, S_i = s)$. The solution to the above differential equation combined with the boundary condition $b_n(0) = 0$ yields a pure strategy, increasing, symmetric equilibrium. By the same reasoning as in Milgrom and Weber (1982), for a given n we can express an equilibrium bidding function as

$$b_n(s) = b_n(s^*(0)) + v_n(s, s) - v_n(s^*(0), s^*(0)) + \int_{s^*(0)}^s L_n(\alpha|s) dt_n(\alpha),$$

where $t_n(\alpha) = v_n(\alpha, \alpha)$ and

$$L_n(\alpha|x) = e^{-\int_{\alpha}^x g_{Y_n|S_i}(s|s)/G_{Y_n|S_i}(s|s)ds}.$$

We first prove the convergence claim for signals that in the limit expect to receive an object with a probability that is strictly between zero and one. That is, we consider signals $s \in (s^*(0), s^*(1))$ and argue that

$$b_n(s) \rightarrow v(s) - \int_{s^*(0)}^s L(\alpha|s) dt(\alpha),$$

where

$$L(\alpha|x) = e^{-\int_{\alpha}^x g_{Y|S_i}(s|s)/G_{Y|S_i}(s|s)ds},$$

$t(\alpha) = v(\alpha)$, $v(s) = \lim_n v_n(s, s) = [s^*]^{-1}(s)$, and where we take L to be 0 whenever the integral diverges.⁵

Given the convergence of $v_n(s, s)$ to $v(s)$, we need only check that

$$\int_{s^*(0)}^s L_n(\alpha|s) dt_n(\alpha) \rightarrow \int_{s^*(0)}^s L(\alpha|s) dt(\alpha)$$

for any $s \in (s^*(0), s^*(1))$.

Consider any $\alpha, x \in (s^*(0), s^*(1))$. We know that $g_{Y_n|S_i}(s|s) \rightarrow g_{Y|S_i}(s|s)$ for any $s > s^*(0)$ (where $g_{Y|S_i}(s|s) = g_{V|S_i}(v(s)|s)$), $G_{Y_n|S_i}(s|s) \rightarrow G_{Y|S_i}(s|s)$ for any $s > s^*(0)$ (where $G_{Y|S_i}(s|s) = G_{V|S_i}(v(s)|s)$), and the $g_{Y_n|S_i}$'s are bounded above uniformly and the $G_{Y_n|S_i}$ are bounded away from zero uniformly (across $s > \alpha$, noting $\alpha > s^*(0)$) by Lemma 1. Therefore, we can apply the Dominated Convergence Theorem to conclude that $L_n(\alpha|s) \rightarrow L(\alpha|s)$. Note that $L_n(\alpha|s)$ is nonnegative and bounded uniformly above (across n) by $e^{-0} = 1$. We also know that $t_n(\alpha)$ converges to $t(\alpha)$ and that these are bounded above uniformly. We can apply the Dominated Convergence Theorem again to conclude that

$$\int_{s_*}^s L_n(\alpha|s) dt_n(a) \rightarrow \int_{s_*}^s L(\alpha|s) dt(a)$$

for any $s \in (s^*(0), s^*(1))$, as claimed.⁶

By the definition of h it follows that

$$h(s) = \frac{\int_{y=0}^s v(y)g_{Y|S}(y|s) dy}{G_{Y|S}(s|s)}.$$

⁵The Y is the limit of the Y_n 's, which becomes a degenerate random variable as a function of V in the limit.

⁶We need to be a bit careful as both the functions and the measures of integration are changing with n . However, both are bounded and converging and so a variation on the Dominated Convergence Theorem can be used. In particular, let $\{h_n\}, \{j_n\}$ be two sequences of functions such that (i) $0 \leq h_n \leq j_n$, (ii) $h_n \rightarrow h, j_n \rightarrow j$ almost everywhere, and let $d\mu_n$ be a sequence of measures that converge set wise to a measure $d\mu$, then $\lim \int j_n d\mu = \int j d\mu$ implies $\lim \int h_n d\mu_n = \int h d\mu$. This follows from a generalized version of Fatou's Lemma that appears in Royden p. 269. This lemma implies that $\liminf \int h_n d\mu_n \geq \int h d\mu$, hence it is enough to show that $\limsup \int h_n d\mu_n \leq \int h d\mu$. Since $j_n - h_n \geq 0$ we can apply again the generalized version of Fatou's Lemma to conclude that $\liminf \int (j_n - h_n) d\mu_n \geq \int (j - h) d\mu$. However, $\liminf \int (j_n - h_n) d\mu_n = \int j d\mu - \limsup \int h_n d\mu_n$. hence, indeed $\limsup \int h_n d\mu \leq \int h d\mu$.

Thus,

$$h'(s) = \frac{g_{Y|S}(s|s)}{G_{Y|S}(s|s)} (v(s) - h(s)).$$

Again (as in Theorem 14 in Milgrom and Weber (1982)) we can write

$$h(s) = v(s) - \int L(\alpha|s) dt(\alpha).$$

This is the same as the expression for $\lim_n b_n(s)$, and thus establishes the claim.

Next, we consider $s > s^*(1)$, note that $s^*(1) < 1$. Since the bidding function is increasing in s_i , it follows that $s^*(1)$ is the asymptotic cutoff conditional on $V = 1$ for getting an object. The MLRP implies that in equilibrium agents are “guaranteed” (with a probability converging to 1 as n grows) to win an object if their signal is above $s^*(1)$. To see this, let b be the inf over $s > s^*(1)$ of $\lim_n b_n(s)$ (for some sequence of convergent subsequences, if necessary). Note that under the MLRP, and given the increasing nature of the equilibrium, the law of large numbers implies that any signal $s > s^*(1)$ is getting an object with a probability approaching 1 in the limit. The claim is that $b_n(s)$ must be converging to b . If not, then there is some $s > s^*(1)$ and $\varepsilon > 0$, such that $b_n(s) > b + \varepsilon$ for infinitely many n . However, by the definition of b , the probability of winning with a bid of $b + \varepsilon/2$ is converging to 1. This contradicts the fact that $b_n(s)$ is a best response for s (for all n), as lowering bids to $b + \varepsilon/2$ would still lead to probabilities arbitrarily close to 1 of winning for large enough n , but would lead to discretely lower payments. This implies that in a discriminatory auction the $b_n(s_i)$ converges to being constant over $[s^*(1), 1]$. That is, $b_n(s'_i) - b_n(s_i) \rightarrow_n 0$ for any $s'_i > s_i > s^*(1)$. ■

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