

Divisible good auctions - the role of allocation rules

Ilan Kremer*

Kjell G. Nyborg**

Abstract

We examine the role of allocation rules in determining the set of equilibrium prices in uniform price auctions. Beginning with Wilson (1979), the theoretical literature has argued that these auctions are subject to possible low equilibrium prices. In this paper we show that this is due to the way the asset is being divided. We focus on allocation rules that specify the way the asset is divided in cases of excess demand. This may have a dramatic effect on the set of equilibrium prices. In particular, we show that the allocation rule used in practice has a negative effect on equilibrium prices.

* Graduate School of Business, Stanford University, Stanford, CA 94305, email: ikremer@stanford.edu.

** London Business School, Regent's Park, London NW1 4SA, UK; and CEPR. email: knyborg@london.edu

We have benefited from comments by Jennifer Reinganum, the editor, two anonymous referees, Asher Wolinsky, Michael Whinston, Peter DeMarzo, Ulf Axelson and seminar participants at the Hebrew University, Tel Aviv

University, Stanford-Berkeley workshop and the 2002 AFA meeting. We would also like to thank Yael Hochberg and Linda Bethel for their help in preparing this manuscript.

1 Introduction

Divisible good auctions are widely used in practice. In a typical auction, bidders submit demand schedules. The auctioneer computes the aggregate demand function and the clearing price. He then divides the asset among several bidders. Previous research (e.g. Wilson, 1979) suggests that divisibility creates a wide range of possible equilibria. In many of these equilibria, bidders are able to maintain non-competitive quotes. This results in low revenues for the seller. Moreover, these equilibria exist even when there is a large number of bidders. In this paper, we arrive at a different conclusion. We argue that this phenomenon is due to the way the asset is being divided rather than the fact that it is divisible. Furthermore, variants of these auctions have been extensively used to describe a wide range of market interactions. We bring into question an implicit assumption made in many of these papers that agents can only use continuous strategies. This assumption is being made, for example, when using calculus of variations to verify that a given strategy profile is indeed an equilibrium. One might guess that since continuous functions can approximate discontinuous functions, this restriction does not alter the set of equilibria. However, we argue that restricting the strategy space to include only continuous functions distorts the set of equilibria, since it forces demand to be equated with supply at the

clearing price.

Once we account for the fact that in practice bidders are allowed to submit discontinuous demand schedules (e.g. limit orders), the question of how to divide the asset arises. There is a potential for discontinuities in the aggregate demand for the asset; that is, the auctioneer might face excess demand at the clearing price. For example, suppose that there are two bidders, Alice and Bob:

Alice demands up to 60% of the asset if the price not higher than \$10 and 50% if the price is not higher than \$20.

Bob demands up to 90% if the price not higher than \$10 and 10% if the price is not higher than \$30.

The clearing price in this example is \$10, as this is the highest price at which all the asset can be sold. However, the auctioneer is facing a demand of 150%. The auctioneer needs to specify ex-ante an “allocation rule” that describes how the asset is divided. Although it is essentially a tie-breaking rule, we show that it has a dramatic effect on equilibrium prices. When discontinuous demand schedules are admissible, we show that by choosing simple allocation rules that satisfy some basic properties, only competitive equilibria exist and the seller maximizes revenues.

The analysis of divisible good auction has gained a lot of interest following the debate of how to auction U.S., Treasury securities. However, the U.S. evidence is that the outcome of these auctions is not extremely sensitive to the exact auction format.¹ Uniform price auctions are also used in other financial markets such as IPOs and share repurchases,² as well as in non-financial markets. An extreme example in which uniform price auctions preformed poorly is the electricity market in California. Uniform price auctions were used to buy electricity on the power exchange. In these auctions, bidders were the suppliers of electricity. During the summer of 2000, suppliers were able to extract the highest possible price in a number of cases. The strategies being used were consistent with the behavior in Wilson (1979). These events led to the collapse of the exchange in March 2001.³

In practice, virtually all uniform price auctions use an allocation rule which gives precedence to demand at a high price. An order at a price higher than the clearing price is guaranteed to be filled and orders at exactly the clearing price are treated on a pro-rata basis. The rationale behind this allocation rule is that a bidder should be rewarded for submitting an order at a high price. While this might look like a rule that encourages bidders to raise their bids, we argue that it may inhibit competition. We show that bidders are not fully rewarded for raising their bids and pushing the market-

clearing price higher. In fact, the rule has the same effect as restricting bidders to submit only continuous demand schedules. This implies that results regarding low revenue equilibria that were obtained in papers such as Wilson (1979) hold here as well. This is because Wilson implicitly assumes that bidders submit continuous demand schedules.

Section 2 describes the basic setup. We define what is an allocation rule and present some basic properties it may have. In section 3 we consider uniform price auctions of a risky asset when bidders are risk neutral. We show that if the rationing rule satisfies some basic properties, the set of equilibrium prices is a singleton. The price extracts all the surplus from bidders. However, if we use ‘bad’ allocation rules or if we restrict bidders to use only continuous demand schedules there exist low equilibrium prices. If bidders are restricted to continuous demand schedules, aggregate demand equals supply and there is no need to specify an allocation rule. We generalize this result by considering the case of a random supply function. We then analyze the allocation rule used in practice and show that it has the same effect as restricting bidders to use continuous bidding strategies.

In section 4 we examine the robustness of our findings. Specifically we consider a discrete state space and private information. We show that low revenue equilibria exist even if the state space is discrete, that is, there ex-

ists a minimal tick size. These equilibria are eliminated with an appropriate allocation rule. We also consider a setup with privately informed bidders. In other parts of the paper, we follow Wilson (1979), Whinston and Bernheim (1986), and Back and Zender (1993) in examining the case in which bidders are identically informed and face a poorly informed seller. While this is a good approximation in some cases (see Whinston and Bernheim, 1986), it is clearly restrictive. The main reason for this is that it enables a full characterization of the equilibria set. With privately informed bidders, we obtain a partial characterization of the equilibria set. As before, low equilibrium prices exist unless a proper allocation rule is used.

In section 5, we consider the case of risk averse bidders. We compare a case in which bidders are restricted to submitting only continuous demand functions to a case in which they are allowed to submit discontinuous demand functions. We obtain an exact characterization of equilibrium prices in both cases. One might suspect that previous results are caused only by the discontinuity of risk-neutral preferences. A risk-neutral bidder demands an infinite amount if the price is below the expected value of the asset and none if the price is above it. However, we show that even with risk averse bidders, allocation rules have a dramatic effect on equilibrium prices. With the allocation rule used in Treasury auctions, there is no lower bound on

equilibrium prices. The upper bound is the price that would prevail if bidders act as price takers. In contrast, with an appropriate allocation rule there exists a lower bound. This lower bound is the price at which a bidder would be indifferent between having the whole asset and having only $1/N$ of the asset, where N is the number of bidders. In the limit, this lower bound converges to the price that is obtained if we sell the object as an indivisible good.

This paper is related to a number of papers. Wilson (1979) is the first to consider uniform price auctions of divisible goods. He shows, in several settings, that there exists an equilibrium in which the asset is sold for half of its value. Klemperer and Meyer (1989) generalize this result by showing that any low price can be supported in equilibrium. They consider a more general setting in which supply is not fixed and is given by a supply function.⁴ They introduce the notion of random supply as a way to reduce the number of equilibria. Back and Zender (1993) examine uniform and discriminatory price auctions. They argue that auctioning treasury notes using a uniform price auction has the potential of yielding very low revenues to the Treasury. They show that while supply uncertainty generated by non-competitive demand reduces the set of equilibria, it does not eliminate underpricing.⁵ More recently, Back and Zender (2001) and McAdams (2002) show that the seller

can reduce, and possibly eliminate, underpricing by choosing a quantity *ex post* that maximizes revenue [see also Hansen (1988) and Lengwiler (1999)]. LiCalzi and Pavan (2002) also study flexible supply and show that the seller can improve revenue by announcing an elastic supply schedule before bidders submit their bids. Unlike our paper, however, these papers study uniform auctions with the standard allocation rule where infra-marginal demand has priority over marginal demand at the clearing price. A more important difference is that the modifications that we examine do not use supply flexibility; the amount being sold is fixed in advance. McAdams (2002) also examines the effect of small amounts of cash awarded to rationed bidders. These amounts may also eliminate underpricing. Our paper differs in that the uniform price format is kept as all bidders pay the same price.

2 The model

A single perfectly divisible common value asset is being sold via an auction to $N > 1$ bidders. The asset's value to all bidders is given by a random variable \tilde{v} , with \bar{v} denoting its the expected value. We assume a simple information structure which is traditionally assumed in this literature (e.g. Wilson (1979), Back and Zender (1993) and Bernheim and Whinston (1986)). An uninformed seller is facing identically informed bidders. In section 4 we

extend the analysis to a setup in which bidders are privately informed.

Bidders submit non increasing demand schedules for the asset; bidders specify what fraction of the asset they demand at every price (per unit)⁶. Let $x_j(p)$ denote bidder j 's demand schedule and $X(p) \equiv \sum_j x_j(p)$ denote the aggregate demand schedule. Let $X_{-i}(p) \equiv X(p) - x_i(p)$ denote aggregate demand when we exclude bidder i 's demand. We assume that $0 \leq x_j(p) \leq 1$ and that $x_j(p)$ is left continuous. If $X(p) < 1$ for all p , we assume that there is no trade, in all other cases, all bidders pay the market clearing price, p^* . We define the market-clearing price to be the highest price at which the whole supply can be sold. Formally, we let:⁷

$$p^* = \max \{p | X(p) \geq 1\}$$

This implies that the seller can sell the whole asset at the market clearing price. However, in many cases there is excess demand at the clearing price, that is, $X(p^*) > 1$. Hence, we define an allocation rule that specifies how the asset is divided in these cases. We require that an allocation rule divides the whole amount in such a way that no bidder gets more than his demand.

Definition 1. An *allocation rule* is a mapping from the set of demand functions $\{x_i(\cdot)\}_{i=1}^N$ to non-negative quantities $\{q_i\}_{i=1}^N \in R_N^+$ so that $\sum_i q_i = 1$ and $q_i \leq x_i(p^*)$ for all i .

Note that since $\sum_i q_i = 1$ and $q_i \leq x_i(p^*)$, allocation rules differ only in situations where aggregate demand at the clearing price exceeds total supply. That is, for every allocation rule, $X(p^*) = 1$ implies that $q_i = x_i(p^*)$ for all i . Our goal is to categorize different allocation rules. For that reason we introduce the following definitions:

Definition 2. An allocation rule is said to satisfy the *majority property* if a bidder whose demand is more than 50% of the aggregate demand gets an amount that is bounded away from 0.5. That is, for every $\alpha > 0.5$ there exists $\beta > 0.5$ such that $x_i(p^*) = \alpha X(p^*)$ implies that $q_i > \beta$.

Definition 3. An allocation rule is said to be *independent of irrelevant demand* if $\{q_i\}_{i=1}^N$ only depends on the demand profile at the clearing price, $\{x_i(p^*)\}_{i=1}^N$.

We continue by giving examples of two allocation rules and apply them to the example we presented in the introduction. In this example the price is set at \$10 and aggregate demand is 150%.

- Allocation rule #1- Pro-rata: Each player gets an amount which is proportional to his demand at the clearing price:

$$q_i = \frac{x_i(p^*)}{X(p^*)}$$

In the example we consider, Alice receives $\frac{60\%}{150\%} = 40\%$ and Bob receives $\frac{90\%}{150\%} = 60\%$.

- Allocation rule #2- Pro-rata on the margin: This allocation rule (which is used in practice) gives a priority to demand at a high price. Demand is first allocated to orders at a price higher than the clearing price. Next, the auctioneer allocates the residual supply to bidders using a pro-rata rule according to the residual demand. Formally this rule is given by:

$$q_j = x_{j+}(p^*) + \frac{x_j(p^*) - x_{j+}(p^*)}{X(p^*) - X_+(p^*)}$$

where $x_{j+}(p^*) = \lim_{p \downarrow p^*} x_j(p)$ and $X_+(p^*) = \lim_{p \downarrow p^*} X(p)$. In the example we consider, Alice receives $50\% + \frac{10\%}{90\%} * 40\% = 54.44\%$ and Bob receives $10\% + \frac{80\%}{90\%} * 40\% = 45.55\%$.

Clearly rule #1 satisfies the majority property and is independent of irrelevant demand while rule #2 satisfies neither property.

3 Risk-neutral and identically informed bidders

In this section we consider the case of risk-neutral bidders. This implies that utility is linear in the amount of the good obtained. The utility from getting

an amount q of the asset at a price p is given by: $q(\bar{v} - p)$. We begin by proving that in equilibrium there is no markdown in the price of the asset.

Theorem 1. If the allocation rule satisfies the majority property, then the unique equilibrium clearing price is $p^* = \bar{v}$.

Proof. We first note that $p^* \leq \bar{v}$, as otherwise any player who gets a fraction of the asset would prefer to deviate and demand zero for $p > \bar{v}$. Assume by contradiction that $p^* < \bar{v}$. We start by assuming that there are more than two bidders. Since $\sum_{i=1}^N q_i = 1$, there is one bidder, say j , for which $q_j < \frac{1}{2}$. Consider the following deviation for bidder j :

$$x_j^{new}(p) = \begin{cases} 1 & \text{if } p \leq p^* + \varepsilon \\ 0 & \text{otherwise} \end{cases}$$

For ε sufficiently small, bidder j would benefit from this deviation. The reason for this is that the new clearing price would become $p^* + \varepsilon$. Since p^* is the original clearing price it must be that $X_{-j}(p^* + \varepsilon) < 1$, and by the majority property bidder j gets more than 0.5 of the asset. Hence, his expected profit will be at least $0.5(\bar{v} - p^* - \varepsilon)$. For ε small enough:

$$0.5(\bar{v} - p^* - \varepsilon) > q_j(\bar{v} - p^*)$$

Now assume that there are exactly two bidders. If $q_1 \neq q_2$, we can repeat the previous argument by letting the bidder who gets the smaller amount

deviate. If $q_1 = q_2 = 0.5$, we start by choosing $\varepsilon > 0$. Since $x_1(p^* + \varepsilon) + x_2(p^* + \varepsilon) < 1$, there is one player, say 1, for which $x_1(p^* + \varepsilon) < 0.5$. Bidder 2, can use the previous deviation, namely:

$$x_2^{new}(p) = \begin{cases} 1 & \text{if } p \leq p^* + \varepsilon \\ 0 & \text{otherwise} \end{cases}$$

and by the majority property bidder 2 gets at least a fraction $\beta > \frac{1}{2}$ of the asset. For ε small enough we have:

$$\beta(\bar{v} - p^* - \varepsilon) > \frac{1}{2}(\bar{v} - p^*)$$

and bidder 2 is better off. *Q.E.D.*

Next we discuss the reason for the differences in this result compared with previous results. For example, Wilson (1979) shows that there exists an equilibrium in uniform price auctions where the asset is sold for half of its value. Wilson constructs a symmetric equilibrium where each bidder demands $x(p) = \frac{1-2p/N\bar{v}}{N-1}$, which implies a clearing price of $p^* = \frac{\bar{v}}{2}$. To show that this is indeed an equilibrium, Wilson considers a possible deviation, say $y(p)$, of player i . As a result of this deviation there will be a new clearing price, say p^0 , and player i 's profit would become $y(p^0)(\bar{v} - p^0)$. Next he computes the optimal deviation, that is he optimizes over all possible demand functions $y(\cdot)$. This optimization is rather complex as he needs to optimize over a set of functions. However, using the market clearing

condition:

$$y(p^0) + (N - 1) \frac{1 - 2p/N\bar{v}}{N - 1} = 1$$

Wilson rewrites player i 's profit as: $2p^0(\bar{v} - p^0)/N\bar{v}$. Hence, instead of optimizing over the set of functions, we can now optimize over a single variable p^0 . This optimization yields the optimal clearing price $p^0 = \frac{\bar{v}}{2}$, and Wilson concludes that this is indeed an equilibrium.

The main difference here is that Wilson assumes that market clearing holds with equality. Hence, each allocation rule assigns bidders with their demand at the clearing price. This assumption can be justified by restricting bidders to submit continuous demand schedules which would imply that at the clearing price, demand equals supply. The existence of such an equilibrium hinges on this assumption. When we consider discontinuous demand schedules these equilibria disappear (including the one described by Wilson). For example, a possible deviation is to place a limit order for the whole asset at a price of $\frac{\bar{v}}{2}$. This would result in getting more of the asset without changing the price.

A simple extension is the case in which there is no fixed amount of supply but rather a random amount that is sensitive to prices. This generalization lets us analyze more general market interactions like the one described in Klemperer and Meyer (1989). Again, with a proper allocation rule, the

auctioneer can eliminate non-competitive outcomes. Specifically, we assume that supply is given by:

$$S_e(p, w) = S(p) + e(w)$$

where $e \in [\underline{e}, \bar{e}]$ is a zero mean random noise which depends on the state w but is independent of p . We assume that $S(p)$ is a non-decreasing function and that the random variable e 's distribution is atomless. The noise term e in the supply function can be viewed in two ways:

- As an uncertainty bidders have regarding the exact supply functions (see Klemperer and Meyer, 1989).
- In the context of Treasury auctions, the noise is due to non-competitive bids (see Back and Zender, 1993).

We define the clearing price as before,

$$p^*(w) = \max \{p | X(p) \geq S_e(p, w)\}$$

Proposition 1. If the allocation rule satisfies the majority property and is independent of irrelevant demand then in equilibrium $p^*(w) = \bar{v}$ with probability 1.

Proof. See the Appendix.

Pro-rata on the margin

Rule #2 is used in virtually all uniform price auctions (e.g. Treasury auctions). Here, the auctioneer first assigns supply to orders which are at a higher price than the clearing price and divides the remaining supply using a pro rata rule. Our main point is that by using this allocation rule the auctioneer implicitly constrains bidders to use continuous demand schedules. That is, when facing any aggregate demand function by the other $N - 1$ bidders, a bidder is not worse off if he restricts himself to use a continuous demand schedule. Formally we argue:

Theorem 2. If pro-rata on the margin is used then (i) continuous demand functions are optimal for any bidder, (ii) any equilibrium which is obtained when restricting bidders to continuous strategies is still an equilibrium.

Proof. (ii) is a direct implication of (i), hence we focus on proving the first part of the theorem. Let A denote the set of price-quantity outcomes that bidder i can achieve, facing a given strategy profile of the other bidders. Define A' in a similar way but restrict bidder i to use only continuous bidding strategies. Clearly $A \supset A'$, we argue that $A = A'$. To see this, denote by $(X_{-i})_+(p)$ the right limit of $X_{-i}(p)$, that is $(X_{-i})_+(p) = \lim_{p' \downarrow p} X_{-i}(p')$. Bidder i 's strategy can be viewed as first choosing a clear-

ing price, p^* , which must satisfy $(X_{-i})_+(p^*) \leq 1$. Using a left continuous demand schedule, bidder i can get at this price any nonnegative quantity $q \in [1 - X_{-i}(p^*), 1 - (X_{-i})_+(p^*)]$. The reason for this is that the other bidders would not get more than their total demand $X_{-i}(p^*)$. This provides us with a lower bound of $\max\{0, 1 - (X_{-i})_+(p^*)\}$. But, bidder i can get any amount q in this range by quoting a continuous demand schedule:

$$x_j(p) = \begin{cases} 1 & \text{if } p \leq p^* + \varepsilon \\ 0 & \text{otherwise} \end{cases}$$

Q.E.D.

Pro-rata on the margin, therefore, can be viewed as a mechanism which restricts bidders to submit continuous demand schedules. Hence, we conclude that the allocation rule is the culprit behind the possibility of ‘bad’ equilibria in Treasury auctions as shown by Back and Zender (1993).

4 Robustness

Discrete versus continuous state space

So far we have described the model in a continuous space. In real life however, prices and quantities are discrete. There is minimal unit (tick) size for both quantities and prices. One may wonder whether low revenue equilibria always disappear once we move to a discrete setup. However, this

is not the case. Consider the following example taken from the introduction of Back and Zender (1993):

An issue of 1-year notes for a total face value of \$10 billion are to be sold and there are three bidders. Each bidder knows that the market yield is 5 percent. Bids are yield-quantity pairs, and these are discrete; the minimal quantity multiple is \$1 million and the minimal yield tick is one basis point or 0.01%. Each bidder demands \$3333 million at 6% and the entire \$10 billion at 20%. Since the market does not clear at 6% the resulting clearing price is (in yield) 20%; a very favorable yield to the bidders but not to the seller. Under the allocation rule used in practice (#2) no bidder wants to deviate. To see this consider the following:

- If the deviation results in a yield that is above 6%, the maximal amount a bidder can get is \$3334 million. Pushing the price even to 19.99% implies losing one basis point on $3333\frac{1}{3}$ million while gaining only $\frac{2}{3}$ million. This deviation is not profitable.⁸
- Pushing the price all the way to 6% is not favorable. Getting \$10 billion at 6% is worse than getting $3333\frac{1}{3}$ million at 20%.⁹

The problem is that under allocation rule #2, orders at 6% take away the incentives to bid aggressively. Agents are not able to get a significant

number of additional units unless the push prices all the way to 6% which would be suboptimal. However, if the allocation rule satisfies the majority property this would not be an equilibrium. By demanding \$10 billion at 19.99% a bidder will get more than \$5 billion at 19.99% which is a profitable deviation. In fact the same argument used in the proof of Theorem 1 shows that:

Proposition 2. Let *tick* denote the minimal price tick and let $N > 2$. If the allocation rule satisfies the majority property, in equilibrium $p^* > \bar{v} - \frac{N}{N-2} * tick$.

Proof. See the Appendix.

Hence, in the example above equilibrium prices would be very close to 5%. More generally, the proposition shows that whenever there are at least 5 bidders, the largest equilibrium underpricing is 1 tick. Kremer and Nyborg (2002) shows that underpricing may also be eliminated under the standard allocation rule by choosing a relatively small tick size for prices and a relatively large quantity multiple.

Private information

We extend our result to the case in which bidders are privately informed. Bidders get signals $\{S_i\}$ that are affiliated with the asset's value, \tilde{v} . The

asset's value is bounded in some interval $[V_L, V_H]$ and signals are bounded in $[S_L, S_H]$. The conditional distribution of s_i given $\tilde{v} = v$ is described by the density function $f(s_i|v)$. We assume that signals are affiliated (see Milgrom and Weber, 1982), that is, $\frac{f(s|v)}{f(s'|v)} \geq \frac{f(s|v')}{f(s'|v')}$ for $s > s', v > v'$. This standard assumption implies that a higher signal implies a higher conditional expected value for the asset.

The bidding strategies may depend on bidders signals and hence are described by $\{x_i(p, s)\}$. We assume that bidding strategies, $x_i(p, s)$, are left continuous with respect to prices p , for any given signal s . The equilibrium price is a random variable that depends on bidders' signals. We again define it as the highest price at which all the asset can be sold:

$$p^* = \sup \left\{ p \mid \sum x_i(p, s) \geq 1 \right\}$$

We first argue that if we restrict strategies to be continuous or if we use allocation rules such as #2 then arbitrary low prices can be supported in equilibrium:

Proposition 3. For any number of bidders and price $p < E(\tilde{v}|s_i = S_L)$, if we restrict strategies to be continuous then there exists an equilibrium in which $p^* = p$ with probability 1.

Proof. See the Appendix.

Proposition 3 implies that the set of equilibrium prices is unbounded from below. Prices may be lower than V_L with probability 1. We know from Theorem 2 that there is a connection between restricting the strategy space to continuous functions and allocation rules such as pro-rata on the margin. So the underpricing here could also be generated by a ‘bad’ allocation rule. However, if the auctioneer uses a ‘proper’ allocation rules then there exists a lower bound on equilibrium prices.

Definition 4. An allocation rule that is independent of irrelevant demand is said to be *strictly monotone* if for any $\varepsilon > 0$ there exists $\delta > 0$ so that $x'_i(p^*) > x_i(p^*) + \varepsilon$ and $x'_j(p^*) \leq x_j(p^*)$ for $j \neq i$ implies that $q'_i > q_i + \delta$.

Note that our canonical example (rule #1) is strictly monotone. For any s let $\underline{v}(s)$ denote the lowest possible value conditional on $s_i = s$, that is, $\underline{v}(s) = E(\tilde{v} | s_i = s, \text{ for all } j \neq i \text{ } s_j = S_L)$. We argue that:

Theorem 3. If the allocation rule is independent of irrelevant demand and strictly monotone, then the event where $p^* < \underline{v}(s_i)$ for more than one bidder occurs with zero probability.

Proof. See the Appendix.

5 Risk aversion

We assume that supply is fixed and bidders preferences are given by an increasing differentiable concave utility function $u(x)$. Without loss of generality, we also assume that $u(0) \equiv 0$. This implies that the bidder's utility when getting a fraction t of the asset at a price p is given by: $U(t, p) \equiv E[u(t(\tilde{v} - p))]$. Using the fact that $u(\cdot)$ is monotone increasing and concave we conclude that:

Lemma 1. (i) $U(t, p)$ is concave in t for every p . (ii) $\frac{\partial U(t, p)}{\partial p} < 0$ for every t .

Proof. See the Appendix.

We argue that the existence of low equilibrium prices depends on which allocation rule is being used. We first show that the maximal equilibrium price is the price we would get if bidders act as price takers. We then show that if 'good' allocation rules are used, then, for a large number of bidders, equilibrium prices in divisible good auctions are at least as good as in indivisible (unit) good auctions. This lower bound is not present if bidders are restricted to continuous demand schedules. We now introduce the following definitions:

Definition 5. p_{pt} is the price that will prevail if all bidders act as price takers,

that is:

$$\frac{1}{N} = \arg \max_t U(t, p_{pt})$$

Definition 6. p_{in} is the price at which a bidder will be indifferent between having the whole asset or just $\frac{1}{N}$ of the asset, that is:

$$U(1, p_{in}) = U(1/N, p_{in})$$

We let $U_1(t, p)$, $U_2(t, p)$ denote the partial derivative of the utility with respect to quantity and price, that is, $U_1(t, p) \equiv \frac{\partial U(t, p)}{\partial t}$ and $U_2(t, p) \equiv \frac{\partial U(t, p)}{\partial p}$.

Definition 7. An allocation rule that is independent of irrelevant demand is said to be *continuous* if the mapping from $\{x_i(p^*)\}_{i=1}^N$ to $\{q_i\}_{i=1}^N$ is continuous.

The rest of this section is organized into three subsections as follows: First, we describe the set of equilibrium prices if bidders are restricted to use continuous demand schedules. Second, we describe the set of equilibrium prices when bidders are allowed to submit left continuous demand schedules. Third, we consider the special case of exponential utility and demonstrate how the set of equilibria changes when allowing for discontinuous demand functions.

Continuous demand schedules

Our main claim in this section is that if we restrict agents to use continuous strategies, or we use the pro-rata on the margin allocation rule, we can support any price p^* which is low enough. By Theorem 2, this would also be the outcome if bidders are allowed to submit discontinuous demand schedules and allocation rule #2 is used. The upper bound on equilibrium prices is the price we would get if bidders act as price takers. We show that this is also a necessary condition for equilibrium price.

Theorem 4. The set of equilibrium prices is given by:

$$P_{1,N} = (-\infty, p_{pt}).$$

Proof. See the Appendix.

As p_{pt} is increasing in the number of bidders, N , an immediate implication of the above theorem is:

Corollary 1. The set of equilibrium prices is increasing in the number of bidders, N , that is, $P_{1,N_1}^* \supset P_{1,N_2}^*$ for $N_1 > N_2$.

Left continuous strategies

In this subsection we revert to our normal assumption that bidders submit left continuous demand functions and that the allocation rule is independent

of irrelevant demand and satisfies the majority property. We also assume that the allocation rule is continuous. We show that in this case there exists a lower bound on equilibrium prices: at equilibrium prices, bidders prefer having $\frac{1}{N}$ of the asset over the whole asset. Together with p_{pt} as an upper bound, we get an exact characterization of equilibrium prices.

Theorem 5. If the allocation rule is independent of irrelevant demand, continuous and satisfies the majority property, then the set of pure strategies equilibrium prices is given by:

$$P_{2,N} = [p_{in}, p_{pt}].$$

As $U(\cdot, \cdot)$ is continuous, we conclude that as $N \rightarrow \infty$, only prices p^* for which $U(1, p^*) \leq 0$ can be supported in equilibrium, that is:

Corollary 2. For large N , equilibrium prices are bounded below by the outcome of an indivisible good auction.

Example

Assume that $\tilde{v} \sim N(\mu, \sigma^2)$ and $u(x) = -e^{-\alpha x}$ for some $\alpha > 0$. In this case:

$$U(t, p) = -E \left\{ e^{-\alpha t(v-p)} \right\} = -e^{-\alpha t(\mu-p) + \frac{1}{2}\alpha^2 \sigma^2 t^2}$$

- Continuous Demand Schedules: We conclude from the previous discussion that any price p which is below p_{pt} can be supported as an

equilibrium price if we restrict bidders to submit continuous demand functions. Hence the set of equilibrium prices is:

$$P_{1,N}^* = (-\infty, \mu - \frac{\alpha\sigma^2}{N}]$$

- Left Continuous Strategies: If we allow bidders to submit discontinuous schedule and use ‘right’ allocation rules, the equilibrium prices are the collections of prices for which $U(\frac{1}{N}, p) \geq U(1, p)$ and are below p_{pt} . Hence the set of equilibrium prices is:

$$P_{2,N}^* = \left[\mu - \frac{N+1}{2N}\alpha\sigma^2, \mu - \frac{1}{N}\alpha\sigma^2 \right]$$

We therefore conclude that $P_{2,N}^* \subset P_{1,N}^*$ which implies that there is less of an underpricing problem if we allow limit orders. Moreover, if we let N go to infinity then in the case where bidders are allowed to submit only continuous demand schedules, bidders still can get very low prices. In contrast, in the case where bidders are also allowed to submit discontinuous demand schedule, prices higher than $p^* = \mu - \frac{1}{2}\alpha\sigma^2$ can be supported. This is the price for which a bidder would be indifferent between having the whole asset and nothing, that is $U(1, p^*) = U(0, 0)$. So we conclude that in the limit as N goes to infinity, divisible good auction are at least as good as indivisible good auctions. The intuition is that risk sharing is improved.

6 Conclusion

This paper has studied the impact of different allocation rules in divisible good, uniform price auctions. We have shown that the low revenue equilibria focused on in the literature exist if bidders are restricted to submit continuous demand functions. However, if bidders are allowed to submit discontinuous functions, such underpricing equilibria do not exist if the allocation rule satisfies the majority property; namely, any bidder whose total demand at the clearing price is more than 50% of aggregate demand will receive more than 50% of supply. When there is supply uncertainty, an additional property is needed; namely that allocations are based only on each bidder's total demand at the clearing price. Allocation rules that satisfy these two properties induce bidders to compete. It is important to note that these are sufficient but not necessary conditions. One can replace the majority property by an appropriate monotonicity property. The basic principle is that bidders should be rewarded for bidding aggressively. The standard allocation rule in uniform auctions violates this principle because infra-marginal demand has priority; thus inhibiting price competition. We have demonstrated the robustness of our findings by considering a discrete strategy space, risk aversion, and private information.

7 Appendix

Proof of Proposition 1. We first argue that $pr(p^*(w) \leq \bar{v}) = 1$. Suppose not, and let W be the set of states in which $p^*(w) > \bar{v}$. Then one bidder, say i , for whom $E(q_i|W) > 0$ will do better by setting $x_i(p) = 0$ for $p > \bar{v}$. We now argue that $pr(p^*(w) < \bar{v}) = 0$. By contradiction assume that $pr(p^*(w) < \bar{v}) > 0$. Let \underline{p}^* denote the lowest equilibrium price, that is $\underline{p}^* = \sup \{p | pr(p^*(w) < p) = 0\}$. Pick $\varepsilon > 0$ small enough and let $A = \{w | p(w) \in [\underline{p}^*, \underline{p}^* + \varepsilon]\}$; clearly $pr(A) > 0$. For every player i , let $q_i|A$ denote the expected amount he gets conditioned on the event A . Since $\sum_i q_i|A \leq S(\underline{p}^* + \varepsilon)$ for some player, say i , we have $q_i|A \leq \frac{1}{N}S(\underline{p}^* + \varepsilon)$. This implies that player i 's profit conditioned on the event A is at most $\frac{1}{N}S(\underline{p}^* + \varepsilon)(\bar{v} - \underline{p}^*)$. Consider now the following deviation for player i :

$$x_i^{new}(p) = \begin{cases} S(\underline{p}^* + \varepsilon) + \bar{v} & \text{if } p \leq \underline{p}^* + \varepsilon \\ x_i(p) & \text{otherwise} \end{cases}$$

As the rule is independent of irrelevant demand, this deviation would have no effect outside the set A . Conditioned on A , bidder i would get more than $0.5S(\underline{p}^* + \varepsilon)$ at a price of $\underline{p}^* + \varepsilon$ and his profit would be at least $0.5S(\underline{p}^* + \varepsilon)(v - \underline{p}^* - \varepsilon)$. For ε small enough we have:

$$0.5S(\underline{p}^* + \varepsilon)(v - \underline{p}^* - \varepsilon) > \frac{1}{N}S(\underline{p}^* + \varepsilon)(\bar{v} - \underline{p}^*)$$

which implies that i benefits from this deviation. *Q.E.D.*

Proof of Proposition 2. There is a bidder whose payoff is at most $(\bar{v} - p^*)/N$.

If he were to demand 1 at a price of p^* plus one tick, his payoff would change to at least $(\bar{v} - p^* - tick)/2$. This is a strict improvement if $p^* < \bar{v} - \frac{N}{N-2} \times tick$.

The proposition follows. *Q.E.D.*

Proof of Proposition 3. Consider the following bidding strategy

$$x_i(p) = \begin{cases} a_1 (p - p^*) + 1/N & \text{if } p \leq p^* \\ a_2 (p - p^*) + 1/N & \text{otherwise} \end{cases}$$

where $a_1 = -\frac{1}{N(N-1)(E(\tilde{v}|s_i=S_L)-p^*)}$ and $a_2 = -\frac{1}{N(N-1)(E(\tilde{v}|s_i=S_H)-p^*)}$. This bidding strategy does not depend on the personal signal. Since $x_i(p)$ is decreasing and $x_i(p^*) = 1/N$, if all agents use this bidding strategy the outcome is a price of p^* and each agent gets $\frac{1}{N}$. The fact that $p^* \leq E(\tilde{v}|s_i = S_L)$ implies that all agents extract positive profits from the auction. We argue that this is an equilibrium. We use a similar argument to that of Wilson (1979). A deviation will result in a new clearing price p^0 in which the deviating agent will get an amount of $1 - (N - 1) x_i(p^0)$. Hence an agent with a signal of s_i considers the following maximization problem:

$$\max q^0 (E(\tilde{v}|s_i) - p^0)$$

where $q^0 = 1 - (N - 1) x_i(p^0)$. Consider two cases:

$p^0 < p^*$ – in this case a deviation results in a lower quantity, q^0 , than the equilibrium quantity of $\frac{1}{N}$. Hence, the difference in the utility is given

by:

$$E(\tilde{v}|s_i) \left(q^0 - \frac{1}{N} \right) + \frac{1}{N} p^* - q^0 p^0$$

We need to show that this difference is negative so this deviation is not profitable. Since $q^0 < \frac{1}{N}$ an upper bound on this difference is

$$E(\tilde{v}|s_i = S_L) \left(q^0 - \frac{1}{N} \right) + \frac{1}{N} p^* - q^0 p^0$$

which is the utility difference an agent with the lowest signal extracts.

The coefficient a_1 is chosen so that an agent with the lowest signals prefers a price p^* to any price $p^0 < p^*$.

$p^0 > p^*$ – in this case a deviation results in a higher quantity, q^0 , than the equilibrium quantity of $\frac{1}{N}$. Again the utility difference is

$$E(\tilde{v}|s_i) \left(q^0 - \frac{1}{N} \right) + \frac{1}{N} p^* - q^0 p^0$$

However, since in this case $q^0 > \frac{1}{N}$, we get an upper bound on this difference of

$$E(\tilde{v}|s_i = S_H) \left(q^0 - \frac{1}{N} \right) + \frac{1}{N} p^* - q^0 p^0$$

which is the utility difference an agent with the highest signal extracts.

Q.E.D.

Proof of Theorem 3. Let \underline{p} denote the lowest equilibrium price, formally:

$$\underline{p} = \sup \{p | p^* > p \text{ with probability } 1\}$$

Suppose $\underline{p} < \underline{v}(s_i)$ and pick a small $\varepsilon > 0$ so that $\underline{p} + \varepsilon < \underline{v}(s_i)$. We argue that $x_i(\underline{p} + \varepsilon, s_i) > 0.5$. This implies the above conclusion as if two agents find this to be true the market clearing price would be strictly above \underline{p} . To see that $x_i(\underline{p} + \varepsilon, s_i) > 0.5$, assume by contradiction that $x_i(\underline{p} + \varepsilon, s_i) < 0.5$ and $\underline{p} + \varepsilon < \underline{v}(s_i)$. Consider the following deviation

$$x_i^{new}(p, s) = \begin{cases} 1 & \text{for } p \leq \underline{p} + \varepsilon \\ x_i(p, s) & \text{otherwise} \end{cases}$$

This deviation has effect only in the case when $p^* \in [\underline{p}, \underline{p} + \varepsilon]$. It is a profitable deviation as conditional on this event. By strict monotonicity it strictly increases the amount i gets with only minimal price impact. *Q.E.D.*

Proof of Lemma 1. Part (ii) of the lemma follows immediately from $u(\cdot)$ being an increasing function. We show now that $U(t, p)$ is concave. This follows from:

$$\begin{aligned} U(\lambda t_1 + (1 - \lambda) t_2, p) &= E[u(\lambda t_1(v - p) + (1 - \lambda) t_2(v - p))] \\ &\geq E[\lambda u(t_1(v - p)) + (1 - \lambda) u(t_2(v - p))] \\ &= \lambda E[u(t_1(v - p))] + (1 - \lambda) E[u(t_2(v - p))] \\ &= \lambda U(t_1, p) + (1 - \lambda) U(t_2, p) \end{aligned}$$

Q.E.D.

Proof of Theorem 4. We first show that any price $p^* < p_{pt}$ can be supported as an equilibrium price. To see this, consider the following symmetric strategy profile with linear demand schedule: $x_i(p) = x(p) = ap + b$ for all i , where $a \equiv \frac{U_2(1/N, p^*)}{(N-1)U_1(1/N, p^*)}$ and $b \equiv \frac{1}{N} - ap^*$. Since $p^* < p_{pt}$ we conclude that: $U_1(\frac{1}{N}, p^*) > 0$ and $U_2(\frac{1}{N}, p^*) < 0$ (Lemma 1). This implies that the demand schedule is decreasing, that is, $a \leq 0$. As $ap^* + b = \frac{1}{N}$ we conclude that the clearing price is p^* . To show that this is indeed an equilibrium we must show that no player i has an incentive to deviate. Since $x(p)$ is continuous, player i can get an amount of $1 - (N - 1)x(p)$ at a price p . Hence, we need to show that:

$$p^* \in \arg \max_p U(1 - (N - 1)x(p), p)$$

Using our definition of $x(p)$, it follows that the first order condition is satisfied. Using the concavity of $U(p, t)$ in t it follows that the second order condition is satisfied. Thus, no player has an incentive to deviate.

We now show that in equilibrium it must be that $p^* < p_{pt}$. Assume by contradiction that p^* is an equilibrium price, and $p^* \geq p_{pt}$. We argue that the bidder who gets the largest amount of the asset has an incentive to deviate by lowering his demand. To see this formally, denote by i the bidder who gets the largest amount, clearly $q_i \geq \frac{1}{N}$. By deviating, player i can get

a quantity of $1 - X_{-i}(p)$ at any price p . His utility in this case is given by $U(1 - X_{-i}(p), p)$. Differentiating this utility yields:

$$\frac{d}{dp}U(1 - X_{-i}(p), p)|_{p^*} =$$

$$-X'_{-i}(p^*)U_1(1 - X_{-i}(p^*), p^*) + U_2(1 - X_{-i}(p^*), p^*)$$

As $p^* \geq p_{pt}$ and $q_i \geq \frac{1}{N}$, we conclude that $U_1(1 - X_{-i}(p^*), p^*) \leq 0$. By Lemma 1 we get that $U_2(1 - X_{-i}(p^*), p^*) \leq 0$ and together with $X'_{-i}(p^*) \leq 0$ ¹⁰, we conclude that bidder i prefers to deviate by lowering the clearing price, and potentially increasing the amount he gets. *Q.E.D.*

Proof of Theorem 5. We start by showing that we can support any $p^* \in P_N$ using a pure strategy symmetric equilibrium. We claim that the following is an equilibrium:

$$x_i(p) = x(p) = \begin{cases} 1 & \text{for } p \leq p^* \\ 0 & \text{otherwise} \end{cases}$$

The reason that this is indeed an equilibrium is that any bidder, say i , can deviate in only two ways;

1. Increase the clearing price by offering to buy the whole asset at some $p > p^*$. In this case i gets the whole asset to himself. But, $U(\frac{1}{N}, p^*) \geq U(1, p^*)$ implies that he is worse off.

2. Get less at the same price, p^* . But again since $U_1\left(\frac{1}{N}, p\right) \geq 0$ bidder i is not better off.

To see that this is a necessary condition let p^* be an equilibrium price.

We consider two possible deviations:

1. Let i denote the bidder who in equilibrium gets the smallest amount, clearly $q_i \leq \frac{1}{N}$. Let $y(p)$ denote the maximal amount i can get at a price p . Since i chooses not to deviate we conclude that,

$$U(q_i, p^*) \geq U(y(p), p)$$

By the majority property we conclude that $y(p) > \frac{1}{2}$ for any $p > p^*$.

Let $y^* = \limsup_{p \downarrow p^*} y(p)$. By continuity of $U(\cdot, \cdot)$ we conclude that $U(q_i, p^*) \geq U(y^*, p^*)$. By the concavity of $U(\cdot, p^*)$ and $q_i \leq \frac{1}{N}$ we conclude that $U\left(\frac{1}{N}, p^*\right) \geq U(1, p^*)$.

2. Let i denote now the bidder who in equilibrium gets the largest amount, clearly $q_i \geq \frac{1}{N}$. By the left continuity of the demand functions and the continuity of the allocation rule we conclude that i could get slightly less at a price not higher than p^* . That is for $\varepsilon > 0$ small enough, i could get $x_i - \varepsilon$ at a price $p \leq p^*$. This implies that for ε small enough, $U(q_i, p^*) \geq U(q_i - \varepsilon, p^*)$. Hence we conclude that $U_1(q_i, p^*) \geq 0$, and by the concavity of $U(\cdot, p^*)$ we conclude that $U_1(1/N, p^*) \geq 0$.

Q.E.D.

REFERENCES

- Back, K. and Zender, J. "Auctions of Divisible Goods: On the Rationale for the Treasury Experiment." *Review of Financial Studies*, Vol. 6 (1993), pp. 733-764.
- "Auctions of Divisible Goods with Endogenous Supply." *Economic Letters*, Vol. 73 (2001), pp. 29-34.
- Bagwell, L.S. "Dutch Auction Repurchases: An Analysis of Shareholder Heterogeneity." *Journal of Finance*, Vol. 47 (1992), pp. 71-105.
- Bernheim D. and Whinston, M. "Menu Auctions, Resource Allocations and Economic Influence." *The Quarterly Journal of Economics*, Vol. 101 (1986), pp. 1-32.
- Hansen, R. "Auctions with Endogenous Quantity." *RAND Journal of Economics*, Vol. 19 (1988), pp. 44-58.
- Kandel, S., O. Sarig, and Wohl, A. "The Demand for Stocks: An Analysis of IPO Auctions." *Review of Financial Studies*, Vol. 12 (1999), pp. 227-247.
- Klemperer P. and Meyer, M. "Supply Function Equilibria in Oligopoly under Uncertainty." *Econometrica* (1989), Vol. 57 (1989), pp. 1243-1277.
- Kremer, I. and Nyborg, K.G. "Underpricing and Market Power in Uniform Price Auctions." Working paper, London Business School, 2002.

- Lengwiler, Y. “The Multiple Unit Auction with Variable Supply.” *Economic Theory*, Vol. 14 (1999), pp. 373-392.
- LiCalzi, M. and Pavan, A. “ Tilting the Supply Schedule Enhances Competition in Uniform-Price Auctions.” working paper, Northwestern University, 2002.
- McAdams, D. “Modifying the Uniform-Price Auction to Eliminate ‘Collusive-Seeming Equilibria’.” Working paper, MIT, 2002.
- Milgrom, P.R. and Weber, R.J. “A Theory of Auctions and Competitive Bidding.” *Econometrica* Vol. 50, pp. 1089-1122.
- Nyborg, K.G. and Sundaresan, S. “Discriminatory versus Uniform Treasury Auctions: Evidence from When-Issued Transactions.” *Journal of Financial Economics*, Vol. 42 (1996), pp. 63-104.
- Wilson, R. “Auctions of Shares.” *Quarterly Journal of Economics*, Vol. 93 (1979), pp. 675-689.

Endnotes

¹Uniform price auctions seem to perform a little better than discriminatory auctions (see for example Nyborg and Sundaresan, 1996).

²See for example Kandel, Sarig and Wohl (1999) for a description of Israeli IPOs and Bagwell (1992) for a description of the use of Dutch auctions in share repurchases

³See http://www.cpuc.ca.gov/published/report/GOV_REPORT.htm for a description of this market.

⁴Klemperer and Meyer (1989) actually study a supply function game with random demand, but their analysis is equally applicable to a demand function game with random supply.

⁵In the primary Treasury security market, the public can participate by submitting non-competitive bids. These bids specify amounts but not prices and are executed at the market-clearing price. Hence, these bids can be thought of as adding a noise term to the available supply.

⁶Most of our results hold even if bidders submit increasing demand functions. However, this assumption simplifies the exposition of many of our results.

⁷Since demand is left continuous, prices are well defined. Note also that prices are unbounded from below. It is straightforward to incorporate a minimum price that will ensure that prices are non-negative.

⁸The original profit is given by $(\frac{1}{1.05} - \frac{1}{1.2}) \frac{10000}{3} = \396.83 million, while the deviation gives a profit of $(\frac{1}{1.05} - \frac{1}{1.1999}) 3334 = \396.67 million.

⁹The deviation gives a profit of no more than $(\frac{1}{1.05} - \frac{1}{1.06}) 10000 = \89.847 million.

¹⁰Formally $X_{-i}(\cdot)$ need not be differentiable. But since it is left continuous around p^* , our analysis is valid.