Underpricing and Market Power in Uniform Price Auctions*

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Abstract

In uniform auctions, buyers choose demand schedules as strategies and pay the same "market clearing" price for units awarded. Despite the widespread use of these auctions, the extant theory shows that they are susceptible to arbitrarily large underpricing. We make a realistic modification to the theory by letting prices, quantities and bids be discrete. We show that underpricing can be made arbitrarily small by choosing a sufficiently small price tick size and a sufficiently large quantity multiple. We also show how one might improve revenues by modifying the allocation rule. A trivial change in the design can have a dramatic impact on prices. Our conclusions are robust to bidders being capacity constrained. Finally, we examine supply uncertainty robust equilibria.

Uniform price auctions are widely used in financial and commodity markets to sell identical securities or goods to multiple buyers. Examples include UK and US Treasury auctions and electricity auctions.[1] In these auctions, bidders compete by simultaneously submitting multiple bids, or demand schedules. The goods are allocated in the order of descending price until supply is exhausted. All bidders pay the same "market clearing" (or stop-out) price. Demand above this price is awarded in full, while marginal demand at the stop-out price is prorated. Uniform auctions thus work much like textbook Walrasian markets, with the notable exception that demand schedules may be submitted strategically.

Given the real world importance of these auctions, it is troubling that the theory shows that they are susceptible to arbitrarily large underpricing. This is a result of market power which arises endogenously; in equilibrium, each bidder is a monopsonist over the supply left over after the other bidders' demand has been filled [Wilson (1979), Back and Zender (1993)].[2] Apart from the pecuniary loss to a seller, underpricing may also give rise to a welfare loss when a potential seller refrains from selling because he expects low revenues.

In this paper, we first show that in a framework that better corresponds to the real world, underpricing may be less severe than what can be concluded from the existing theory (e.g. Wilson/Back and Zender). Specifically, we incorporate discrete bids into the model. That is, bidders submit multiple price-quantity pairs rather than explicit demand functions. There may also be a tick size for prices, a quantity multiple, and a limit on the number of bids allowed. We show that equilibrium underpricing can be made arbitrarily small by an appropriate choice of tick size and quantity multiple. This may help shed light on the prevalence of uniform auctions in practice despite the theoretical warnings of severe underpricing.

We then show that underpricing can also be eliminated by modifying the way supply is allocated when there is excess demand. In contrast to the standard allocation rule which gives preference to demand above the stop-out price, our modification gives this inframarginal demand only partial preference. There are several reasons to consider alternative allocation rules. First, excess demand is a feature of most auctions in practice due to bidders submitting discontinuous demand schedules. Second, alternative allocation rules may eliminate some equilibrium underpricing that is possible under particular price ticks and quantity multiples. Third, even though equilibrium underpricing can be made arbitrarily small under the standard allocation rule and discreteness, it is possible that underpricing can be almost an equilibrium [i.e. an epsilon-equilibrium (Radner, 1980)]. In this case, alternative allocation rules may improve revenues by giving bigger rewards to aggressive bidders. The change in design we consider takes into account that players may be capacity constrained and unable to absorb the entire supply. However, the design maintains the uniform auction pricing rule that winners pay the stop-out price. The basic ingredients of the uniform price mechanism are therefore kept intact. From a normative perspective, our results show that the micro-design of a uniform price auction can dramatically improve its performance.

The main idea behind these two results is that the degree of underpricing is determined by the tradeoff a bidder is facing between a price improvement and an increase in his allocation. In the Wilson/Back and Zender underpricing equilibria, bidders are reluctant to bid aggressively because they cannot increase their allocations substantially with only a minor change in the price. Our modifications create a "bigger bang for the buck"; a bidder can get a substantial increase in his allocation with only a modest price increase. This is similar to the difference between Cournot and Bertrand competition: the competitive outcome arises under Bertrand competition because firms realize that by even a slight change in price they can capture the entire market.

Our first result follows from the fact that under quantity discreteness, there is usually a non-negligible residual supply (the supply left over after demand above the stop-out price is awarded). A bidder can increase his allocation with only a minute price change. Thus, discreteness creates price competition for the marginal units. When there is no tick size, we show that this marginal price competition will ratchet up the stop-out price until underpricing is eliminated, even if there is a cap on the quantity that a single bidder can buy and the supply is uncertain. When there is a tick size, we show that the maximum equilibrium underpricing is decreasing in the number of bidders and the quantity multiple, and increasing in the auction size. Intuitively, this is because these parameter changes tend to make the margin large relative to a bidder's allocation, making it more attractive to bid up the price to capture the residual supply.

To see the intuition behind our second result, imagine that there are only two bidders, Alice and Bob. Alice places a bid at a price that exceeds the asset's value. While this need not have an effect on the price she pays, it does affect the allocation. Since Bob will lose money by pushing the stop-out price above the asset's value, Alice's high bid will be awarded to her in full under the standard allocation rule. Bob is therefore limited in the amount he can receive. Thus he faces a low tradeoff between price and allocation. He would face a steeper tradeoff if Alice's high bid received only partial preference by the allocation rule. In sum, awarding demand above the stop-out price in full has the negative effect of inhibiting competition because it limits the rewards from bidding more aggressively.

Finally, we show that among the plethora of equilibria in the basic Wilson/Back and Zender model, there exists a unique class of symmetric supply uncertainty robust underpricing equilibria. These were first discussed by Back and Zender (1993). Our first contribution here is proving existence and uniqueness without exogenously imposing any regularity conditions on the admissible set of demand schedules. For example, the differentiability of the supply uncertainty robust schedules is derived rather than assumed. This is a useful result for empirical research using individual bidder data and aiming to examine the monopsonistic market power theory, because it provides a natural set of equilibria to focus on when there is supply uncertainty (see Keloharju, Nyborg, and Rydqvist, 2003). Our other contribution is showing that in the discrete version of the Wilson/Back and Zender model, there is no supply uncertainty robust underpricing equilibrium. Hence, once again discreteness is seen to weaken the underpricing result of the extant theory.

The empirical evidence shows that the severe underpricing in uniform auctions suggested by Wilson (1979) and Back and Zender (1993) usually does not materialize in practice. For example, in an early study of the US experiment with uniform auctions in the 1990s, Nyborg and Sundaresan (1996) document underpricing (in yields) of a fraction of a basis point. Moreover, they show that this compares well with underpricing under the discriminatory format. This is noteworthy, in part because monopsonistic market power is not an issue in discriminatory auctions (Back and Zender, 1993). Using a larger and updated sample, Goldreich (2003) finds similar results. The US Treasury's own internal study led to the same conclusion and to a complete switch from discriminatory to uniform auctions for selling treasury securities. This was summed up by Larry Summers, then the Deputy Secretary of the US Treasury: "The findings indicate that uniform-price auctions can allow the Treasury to make improvements in the efficiency of market operations and reduce the costs of financing the Federal debt."[3]

In other work, Feldman and Reinhart (1996), Umlauf (1993), and Tenorio (1993) find that uniform auctions have performed better than discriminatory auctions in IMF gold auctions, Mexican treasury auctions, and Zambian foreign exchange auctions, respectively. However, Simon (1994) presents evidence that uniform auctions performed worse than discriminatory auctions in a US Treasury auction experiment in the 1970's. An experimental study by Goswami, Noe, and Rebello (1996) show that when underpricing equilibria actually exist, subjects are able to reach them, particularly if pre-play communication is allowed. Finally, Keloharju, Nyborg, and Rydqvist (2003), using individual bidder data from Finnish Treasury auctions, find little evidence that bidders enjoy monopsonistic market power.

This paper relates to a growing theoretical literature on monopsonistic market power in uniform auctions. For example, Back and Zender (2001), McAdams (2002), and Pavan and LiCalzi (2002) show that flexible supply can reduce underpricing in the Wilson/Back and Zender model. Kremer and Nyborg (2002) also study alternative allocation rules as a means to reduce underpricing, but do not consider capacity constrained bidders. We show here that such constraints have a significant impact on optimal design. Rationing is also studied in a private value, discriminatory auction model by Gresik (2001). See Klemperer (1999) for a more general survey of auction theory.

The rest of the paper is organized as follows. Section 1 describes the basic model, and Section 2 provides analysis. Section 3 introduces discreteness. Section 4 considers alternative allocation rules in uniform price auctions. Section 5 studies supply uncertainty robust equilibria in the basic model. Section 6 provides concluding remarks. The appendix contains proofs.

1 Model

We study the same basic model as Wilson (1979) and Back and Zender (1993). A seller holds an auction of a possibly random quantity $\tilde{Q}$ of a homogenous and perfectly divisible good. The distribution of the auction size $\tilde{Q}$ has support *in* $[0, Q_{\max}]$, $Q_{\max}$ is the least upper bound of the support, and the probability that $\tilde{Q} = 0$ is zero. This fits a scenario where the quantity put up for sale is $Q_{\max}$, but where this can be reduced exogenously.[4] Realizations of $\tilde{Q}$ are denoted by Q.

There are N risk neutral bidders, who compete by simultaneously submitting non-increasing demand functions. The per unit value of the good to each bidder is v and is common knowledge.[5] The reservation price is 0, and a single bidder cannot demand more than $\overline{Q}$. Hence, a strategy for bidder n is a non-increasing function $x_n : [0, \infty) \to [0, \overline{Q}]$. Let $\mathbf{x} = \{x_n\}_{n=1}^N$ denote a profile of strategies for the N bidders, and let $X(p) = \sum_{n=1}^N x_n(p)$ be the aggregate demand function.

Let q_n denote the quantity awarded to bidder n and let p_0 denote the stop-out price, which is the price bidders pay per unit of award. Bidder n's payoff is

$$\pi_n = (v - p_0) \times q_n, \tag{1}$$

and his problem is to choose a demand schedule to maximize the expected value of π_n. Note that q_n and p_0 are endogenous and depend on Q and all bidders' strategies. Specifically, the stop-out price is

$$p_0(Q, \mathbf{x}) = \sup \{p | X(p) \geq Q\}. \tag{2}$$

Alternatively, if demand is lower than supply, $X(p) < Q$, for all prices, the stop out price is the reservation price, that is, $p_0 = 0$.

When there are multiple market clearing prices ($X(p) = Q$), (2) picks the highest one as the stop-out price. When there is no market clearing price because of a discontinuity in the aggregate demand function, the stop-out price is the point of the discontinuity. In this case, if the aggregate demand schedule is left continuous, there is excess demand; and if it is right continuous, there is excess supply. Given p_0, the allocation rule is

$$q_n(p_0; Q, \mathbf{x}) = \begin{cases} x_n(p_0) & \text{if } R(p_0) \geq 0 \\ x_n^+(p_0) + R(p_0)\frac{dx_n(p_0)}{dX(p_0)} & \text{otherwise,} \end{cases} \tag{3}$$

where $x_n^+(p_0)$ is the demand above the stop-out price, that is $x_n^+(p_0) = \lim_{p \to p_0^+} x_n(p)$; $dx_n(p_0)$ is the marginal demand at the clearing price, that is, $dx_n(p_0) = x_n(p_0) - x_n^+(p_0)$; and $R(p)$ is the residual supply at p, that is $R(p) = Q - X(p)$. The top expression in (3) says that if there is no excess demand at the stop-out price, the quantity awarded to bidder n is simply his demand at this price, $x_n(p_0)$. The bottom expression says that if there is excess demand, a bidder will receive all his inframarginal demand, $x_n^+(p_0)$, plus a prorated portion of his marginal demand at p_0. In this case, note that if supply is almost exhausted at a price marginally above p_0, i.e. $X^+(p_0) = Q$, then the quantity that is being rationed has measure zero. So the second term in the bottom expression of (3) would not add anything to q_n. This rationing term only becomes significant when $X^+(p_0) < Q$.

Equation (3) describes the standard allocation rule, which is also what is studied by Back and Zender (1993). Demand above the stop-out price is awarded in full, while marginal demand at the stop-out price is prorated. As we shall see in Section 4, this prioritization of inframarginal demand may reduce competition as compared with an allocation rule that relaxes the priority of high bids.

All aspects of the model are common knowledge among the bidders. The focus is on Nash equilibria in pure strategies. Sections 2 and 5 concentrate on symmetric equilibria, but Sections 3 and 4 also consider asymmetric equilibria. An equilibrium is referred to as an "underpricing equilibrium" if it results in $p_0 < v$ with positive probability. When $\overline{Q} > \frac{Q_{\max}}{N-1}$, there is an equilibrium without underpricing; for example, each bidder submits his Marshallian demand schedule, where $x_n(v) = \overline{Q}$.

In later sections, the model will be modified in the following ways: Discreteness in demand schedules is introduced in Section 3 and alternative allocation rules are introduced in Section 4.

2 Underpricing Equilibria

This section serves several purposes. First, it provides an introduction to the underpricing equilibria of Wilson (1979) and Back and Zender (1993) and thereby sets the stage for the discreteness and rationing analysis that follows. Second, it introduces a simple program for computing a plethora of differentiable underpricing equilibria, without actually imposing differentiability exogenously. This is useful because the explicit construction helps clarify how the underpricing equilibria work and why some equilibria are robust to supply uncertainty while others are not. The methodology can also be extended to risk averse bidders (Keloharju, Nyborg, and Rydqvist, 2003). Third, the analysis of this section forms the basis of the uniqueness proof in Section 5, since no regularity condition on admissible demand functions is imposed exogenously.

It is initially assumed that the auction size is Q with certainty and that $\overline{Q} = \infty$. Supply uncertainty and restrictions on $\overline{Q}$ are discussed towards the end of the section. The focus in this section is on symmetric equilibria, so it is assumed that bidders $2, \ldots, N$ submit the same demand schedule, $x(p)$, which is *strictly* decreasing and differentiable, with $(N-1)x(v) < Q$ and $x(0) \geq Q/N$.[6] The objective is to find $x(p)$ such that bidder 1's best response is also $x(p)$. Since the set of demand functions bidder 1 can choose from is larger than the set of differentiable and strictly decreasing functions, the following result needs to be established before proceeding with the analysis:

Lemma 1 *Bidder 1 can do no better than submitting a demand schedule which is differentiable and strictly decreasing on* $[0, v]$ *and which is such that aggregate demand equals supply at the stop-out price,* p_0*.*

Denote by y bidder 1's demand schedule so that $y(p)$ represents his demand at a price p. Given the demand of the other bidders, x, and the total supply, Q, we can think of the stop-out price, p_0, as a function of y. Lemma 1 implies that bidder 1's problem can be written

$$\max_{y} \left[v - p_0(y)\right] y(p_0) \tag{4}$$

$$\text{such that } y(p_0) + (N-1)x(p_0) = Q. \tag{5}$$

Using Wilson's (1979) insight that picking a demand schedule in this case can be reduced to the much simpler problem of picking a profit maximizing stop-out price, bidder 1's problem can be written as

$$\max_{p_0} (v - p_0)[Q - (N-1)x(p_0)]. \tag{6}$$

That is, the bidder's payoff is given by the product of the profit per unit (or the under-pricing) and the residual supply. As the bidder lowers p_0, he increases profit per unit but lowers the share he receives (the residual supply). This tradeoff is essentially the same as that faced by a monopsonist. The resulting market power is exercised by choosing a stop-out price below v. The first order condition of (6) is given by:[7]

$$(N-1)(v - p_0)x'(p_0) + Q - (N-1)x(p_0) = 0. \tag{7}$$

In a symmetric equilibrium, $y = x$ and, by (5),

$$Nx(p_0) = Q. \tag{8}$$

By substituting this into the first order condition, a multitude of symmetric underpricing equilibria can be found. Here, we present two classes of equilibria, which are interesting because they have previously been studied by Wilson (1979) and Back and Zender (1993).[8]

Proposition 1 *Suppose the auction size is* Q *and* $\overline{Q} = \infty$. *(i) There is a class of symmetric* **linear equilibria** *given by*

$$x(p) = \frac{Q}{N} + \frac{\gamma Q}{(1-\gamma)N(N-1)} - \frac{pQ}{(1-\gamma)vN(N-1)}. \tag{9}$$

The stop-out price is γv, *where* γ *can be any number between 0 and 1.*[9] *(ii) (Back and Zender, 1993) There is a class of symmetric* **nonlinear equilibria** *given by*

$$x(p) = a(1 - p/v)^{\frac{1}{N-1}}, \tag{10}$$

where $a \geq Q/N$. *The stop-out price is*

$$p_0 = \left(1 - \left[\frac{Q}{aN}\right]^{N-1}\right) v, \tag{11}$$

which, depending upon a, *can be any number between* 0 *and* v.

The basis of the underpricing equilibria in Proposition 1 is that as a result of the decreasing demand function submitted by all other bidders, each bidder is a monopsonist over the residual supply. Moreover, the resulting market power is optimally exercised by choosing the same decreasing demand function as the others, so that the stop-out price is below v. This cements the equilibrium since the other bidders therefore find it optimal to submit the decreasing demand function in the first place. A bidder's incentive to bid more aggressively in order to capture a larger share is offset by the "large" price increase required to do so. Similarly, a bidder's incentive to bid less aggressively in order to reduce the stop-out price is offset by the resulting reduction in his share. So in equilibrium, the bidders are essentially in a sort of "complicit agreement" to give each other market power and thereby create underpricing.

The linear equilibria in Proposition 1 have been derived by requiring the first order condition (7) to hold only at the stop-out price. The nonlinear equilibria have been derived by requiring the first order condition to hold at every price. This is why the Wilson-style linear equilibria are sensitive to Q, whereas Back and Zender's nonlinear equilibria are independent of Q. *As a consequence, when supply is uncertain, the nonlinear equilibria of Back and Zender, with* $a \geq Q_{\max}/N$*, will always constitute equilibrium irrespective of the distribution of supply.* Additionally, Back and Zender (1993) argue that: "As in Klemperer and Meyer (1989, Section 3) one can show, under some mild regularity conditions, that the [nonlinear equilibria in Proposition 1] are the unique symmetric pure strategy equilibria having the property that bidders' demand curves 'pass through ex-post optimal points'." (Back and Zender, 1993, p.754). With respect to these regularity conditions, Klemperer and Meyer's approach assumes differentiability. They also rule out non-existence of market clearing prices or the existence of multiple market clearing prices by assuming that such situations would give the players zero profits. In Section 5, we extend the current analysis and generalize Back and Zender's uniqueness result without imposing any regularity conditions on demand functions and without Klemperer and Meyer's "zero profit" assumption.

We close this section by noting the role of $\overline{Q}$ in the analysis above. A finite $\overline{Q}$ would restrict $x(0)$ and therefore (9) and (10) would not constitute equilibria for "large" γ and a, respectively. The "fix" is simple: For $\overline{Q} > \frac{Q}{N}$, let $x(p)$ be given by (9) or (10) and let

$r = \max\{\, p \,|\, x(p) \geq \overline{Q}\,\}$. Then z satisfying the following is an equilibrium: $z(p) = x(p)$ for $p \geq r$ and $z(p) = \overline{Q}$ otherwise. Thus, since any $p_0 < v$ is still an equilibrium stop-out price, the role of $\overline{Q}$ is economically quite insignificant in this model.

3 Demand Schedule Discreteness

Unlike the "smooth" equilibrium demand schedules studied so far, demand schedules in practice are usually discrete. Bidders are typically asked to submit not functions, but price-quantity pairs as bids. Furthermore, since bids must normally be specified in numeric form, it is physically impossible to submit an infinite number of them. Also, there is usually a tick size for prices and a quantity multiple. For example, in UK Treasury auctions, prices must be in increments of 1 pence per £100 of face value; and the quantity for a single bid must be a multiple of £1 million of face value.[10] In some cases, there is a bound on the number of bids that can be made. In Italian Treasury auctions, for example, bidders are limited to 3 bids each (Scalia, 1996).

In this section we study how discreteness affects underpricing. We first consider the case that bidders can only make a finite number of bids, without making additional restrictions. We then study the impact of having both a tick size and quantity multiple. The analysis and results of this section are general in that they place no restrictions on the symmetry of equilibria.

3.1 Finite Number of Bids (Quantity Discreteness)

We assume that bidders choose finite collections of bids as strategies. Each bid is a price-quantity pair, (p, q), representing the marginal demand at a given price. Denote the set of bids submitted by the nth bidder by

$$\boldsymbol{b}_n = \{(p_{n,i}, q_{n,i})\}_{i=1}^{T_n}, \tag{12}$$

where $T_n < \infty$ is the number of bids. As a result, demand is divided up into discrete units, albeit not in a predetermined way. Each bidder can split the total quantity he bids for any way he chooses. Note that the rules of the auction may or may not specify a maximum number of bids (maximum T_n); our results hold in either case. The collection of bids (12) can be represented by a left continuous monotonically decreasing step function as follows:

$$x_n(p) = \sum_{i=1}^{T_n} q_{n,i} 1_{[p_{n,i} \geq p]}. \tag{13}$$

Hence, restricting bidders to make a finite number of bids is equivalent to restricting them to use left continuous monotonically decreasing step functions with a finite, but not necessarily bounded, number of steps. Our main result is as follows:

Theorem 1 *Suppose that each bidder can submit only a finite number of bids. Let* $Q_{\max}$ *be the least upper bound of the support of the distribution of the auction size and suppose that* $\overline{Q} > Q_{\max}/(N-1)$. *The equilibrium stop-out price is* v *almost surely.*

This no-underpricing result stands in sharp contrast to the underpricing results in Proposition 1. The intuition for why allowing only a finite number of bids eliminates underpricing relates to the fact that this restriction makes the margin "large." That is, for any realization of supply, the amount left over after inframarginal bids have been filled, $R^+(p_0)$, has positive mass. Therefore, a bidder can capture a "large" increase in quantity by a negligible increase in price, with positive probability. Price competition over marginal units is thereby encouraged, with the result that the seller obtains the true value of the objects even when supply is uncertain or $\overline{Q} < Q_{\max}$.

For a heuristic proof, note first that for a given demand profile which results in underpricing, there is a lowest stop-out price, which occurs with positive probability. Bidders therefore engage in price competition for the marginal quantity at this lowest stop-out price. This increases the lowest stop-out price to the penultimate lowest stop-out price, and so on. Thus, price competition for the marginal units at the lowest stop-out price ratchets up this price until it equals v.

The reason why the price competition argument does not eliminate underpricing in the original model of Wilson and Back and Zender is that in their underpricing equilibria the quantity left over to marginal demand at the stop-out price has measure zero. This type of construction is possible because there are few restrictions on the set of admissible demand functions. For example, (10) is continuous and therefore supply is almost exhausted by the inframarginal bids, that is $Q - X^{+}(p_0) = 0$. In this case, bidders obviously have no incentives to compete aggressively for the marginal units.

The current analysis illustrates that the reason there is underpricing in Wilson's and Back and Zender's model derives from the fact that the allocation rule gives priority to inframarginal demand, while marginal demand is rationed. Furthermore, in their underpricing equilibria, the amount that is being rationed is insignificant – it has zero measure. Thus bidders have no incentives to compete for the marginal quantity.[11]

3.2 Tick Size and Quantity Multiple

This subsection considers the situation that one normally sees in practice; namely that there is a tick size, $h > 0$, so that prices must be multiples of h, and a quantity multiple, $w > 0$, so that quantities must be multiples of w. Intuitively, the larger the tick size, the more costly it is for a bidder to raise the stop-out price. A large tick size may therefore discourage competition. However, one can show that if only one of these restrictions is imposed then the only outcome is the competitive one, $p_0 = v$.[12] We focus on the more realistic case where both restrictions are imposed.

The presence of a quantity multiple implies that the number of bids any bidder can make is bounded by $\overline{Q}/w$. However, the exact number of bids agents are allowed plays no role in the analysis. We only assume it is at least two. For simplicity, we also assume that:

- A1 - There is no supply uncertainty. Additionally, the auction size, Q, is divisible by w.
- A2 - v is divisible by h.
- A3 - $\overline{Q}$ is divisible by w. $\overline{Q}/w \geq 2$ and $\overline{Q} > Q/(N-1)$.

Assumptions A1 and A2 are not crucial but help simplify the exposition. Assumption A3 has more substance as it implies that no bidder is pivotal. That is, removing any bidder still leaves enough potential demand to cover the auction.

Back and Zender (1993) and Goswami, Noe, and Rebello (1996) present numerical examples in which equilibrium prices are well below v. Here, we generalize their analysis and show that there is an underpricing bound which is linked to the number of bidders, the tick size, and the quantity multiple. In the absence of supply uncertainty, this dependence on the number of bidders is a unique feature of the discrete setup.

Theorem 2 *Assume that there is a tick size, h, a quantity multiple, w, and A1-A3 hold. There is an integer $t^* \geq 1$ such that $p_0 = v - t_0 h$ is an equilibrium stop-out price for integers $t_0 \in \{0, 1, \ldots, v/h\}$ if and only if $t_0 \leq t^*$. We can bound t^* by:*

$$\max\left\{1, \frac{Q}{(N-1)w}\right\} \leq t^* \leq \frac{Q}{(N-1)w} + 1. \tag{14}$$

This result expands on the extant literature by providing a general formula for equilibrium underpricing, which is easy to compute. The intuition as to why there is an underpricing bound relates to the tradeoff between price and quantity. In any underpricing equilibrium, the number of units awarded to bids placed at the stop-out price must be exactly w in total. For if it were more, a bidder could increase his allocation without increasing the stop-out price. In contrast, when it is w, a bidder can only increase his allocation by increasing the stop-out price by at least one tick. His incentive to do so is stronger if the profit he gains on the units he wins is large relative to the loss he suffers on his inframarginal units from the raised stop-out price. In other words, a bidder's incentive to capture the final w units by more aggressive bidding is increasing in the degree of underpricing. As a result, arbitrarily large underpricing cannot be sustained in equilibrium.

The proof of the theorem also shows that the equilibrium with the largest underpricing occurs when all N bidders share in the final w units in equal fashion. For example, if $(Q-w)/N$ is a multiple of w, the equilibrium with the highest underpricing is one where, for all i, $x_i(p) = (Q-w)/N$ for $p \geq p_0 + h$ and $x(p_0) = \overline{Q}$. This is as in Goswami et al (1996). But Theorem 2 also considers the more general case that $(Q-w)/N$ is not a multiple of w, in which case equilibrium turns out to be asymmetric.[13] Focusing on the equilibria with the largest underpricing, some implications of the theorem are:

- Equilibrium underpricing is decreasing in the number of bidders and the quantity multiple.

- Equilibrium underpricing is increasing in the auctioned quantity and the tick size.

- t^* is independent of the tick size. Therefore, regardless of the values of w and the other parameters, underpricing can be made arbitrarily small by making the tick size, h, arbitrarily small.

As mentioned above, an interesting implication of Theorem 2 is that removing the restriction on the price tick size while keeping the quantity multiple eliminates underpricing. That is, when $h = 0$ and $w > 0$ there is no underpricing, essentially since the maximal underpricing measured in ticks, t^*, is independent of the tick size. It is also true that underpricing can be eliminated by keeping a tick size, $h > 0$, but removing the restriction on quantities ($w = 0$). For in this case, underpricing would create an opportunity for a bidder to increase his allocation without changing the price. In particular, since the residual supply will have positive measure (due to discreteness), a bidder could always get a larger portion of this by demanding a fraction of it one tick above the stop-out price. Thus underpricing is incompatible with equilibrium. This illustrates that removing one restriction, but not both, completely eliminates equilibrium underpricing.

As an example, consider a Treasury note auction where the total face value is \$10 billion and each t-note has a face value of \$1,000. Thus $Q = 10$ million. Suppose also that the quantity multiple is \$1 million of face value, which translates into $w = 1,000$ t-notes. Finally, suppose there are 11 bidders. Then, by (14), $t^* \leq \frac{10^7}{10 \times 10^3} + 1 = 1001$. Therefore, if $h = \$0.01$ the underpricing per t-note is bounded above by approximately \$10, or 1% of face value. Empirically, this would be considered a "large" underpricing, but it is far from the arbitrarily large underpricing suggested by the extant literature. The underpricing can be reduced to approximately \$1, or 0.1% of face value, by either increasing the quantity multiple to 10,000, increasing the number of bidders to 101, or decreasing the tick size to \$0.001. The general message that emerges is that underpricing is sensitive to the choice of the quantity multiple, the tick size, and the number of bidders. By choosing a small tick size and a large quantity multiple, a seller can ensure almost competitive pricing in a uniform price auction in the perfect information setting studied here. This also relates to the no-underpricing result in Theorem 1, where bidders can increase their allocations at an arbitrarily small increase in price.

Our example also relates to the discrete example in Back and Zender (1993). They show in the context of the example above, when $N = 3$ and the tick size is 1 basis point, that t-notes with a secondary market yield of 5% can be sold in the auction at a yield of 20%. Note that they consider a tick size in yield while we discuss a price tick.[14] However, our analysis can easily be modified and the conclusions are the same. A 1bp yield tick corresponds to approximately a \$0.1 price tick per \$1,000 of face value. If there were 11 bidders, then a yield of 20% would not be an equilibrium outcome. The reason is that bidders get a smaller allocation and the relative value of obtaining an extra unit is higher, as compared with $N = 3$. As seen above, if in addition the price tick were \$0.01 per \$1,000 of face value (equivalent to a yield tick of approximately 0.1 basis points), the largest equilibrium yield would be approximately 1 percentage point above the market yield, or 6%.

4 Alternative Allocation Rules

In this section, we push the normative analysis further by showing how one might design a uniform auction to improve revenue. Our approach does not involve changing the amount sold. This distinguishes our analysis from that of Back and Zender (2001), McAdams (2002), and Pavan and LiCalzi (2002), who study mechanisms which rely on commitments by the seller to vary supply according to some predefined rule.

Our focus is on a relatively ignored detail, namely rationing; i.e. how supply is allocated in the case of excess demand. We saw above that the standard allocation rule, (3), promotes price competition only for the marginal units, since it discriminates completely in favor of inframarginal demand. This can lead to underpricing when the marginal units are "few". In addition, even if the tick size and quantity multiple are such that Nash equilibrium underpricing is (almost) eliminated, there can be epsilon-equilibria (Radner, 1980) where underpricing is significant. If a bidder requires an increase in his profit of $\varepsilon > 0$ to improve the price by one tick, he may not find it worthwhile to do so if the increase in his allocation would be only a fraction of the quantity multiple. As we suggest in the introduction, a more competitive outcome may arise when we improve the tradeoff a bidder is facing between a price improvement and an increase in his allocation. Changing the allocation rule to give less priority to inframarginal demand may achieve this.

We define an allocation rule to be a mapping from the set of demand functions $\{x_n(\cdot)\}_{n=1}^N$ to non-negative quantities $\{q_n\}_{n=1}^N \in R_N^+$ so that: (i) if $X(p_0) > Q$ then $\sum_n q_n = Q$ and, for all n, $q_n \leq x_n(p_0)$; and (ii) if $X(p_0) \leq Q$ then $q_n = x_n(p_0)$. Thus allocation rules differ only in situations where aggregate demand at the stop-out price exceeds total supply. When demand is less than or equal to supply, all allocation rules yield the same outcome.

Our analysis builds on Kremer and Nyborg (2002). We expand on the analysis in that paper by considering capacity constraints ($\overline{Q} < Q_{\max}$). Such constraints could be built into the auction by design – for example, in US Treasury auctions, $\overline{Q} = 35\%$ – or from bidders' limitations. As we shall see, these constraints are important in designing the allocation rules. We initially revert to the original assumption of the basic model in Section 1 that a bidder can submit any monotonically decreasing demand function, but discuss discreteness towards the end of the section. As in Section 3, the analysis in this section allows for asymmetric equilibria. We take the auction size, Q, as given, but will also discuss the impact of supply uncertainty.

When $\overline{Q} \geq Q$ and bidders can submit continuous demand functions, Kremer and Nyborg (2002) show that there is no equilibrium underpricing if the allocation rule satisfies the following property:

Majority Property - when $X(p_0) > Q$, a bidder whose demand at the stop-out price is more than 50% of aggregate demand gets an amount that is bounded away from $0.5Q$. More precisely, for every $\alpha > 0.5$ there exists $\beta > 0.5$ such that when $X(p_0) > Q$ then $x_n(p_0) = \alpha X(p_0)$ implies that $q_n > \beta Q$.

The standard allocation rule (3) does not satisfy the majority property. An example of a rule that does is the *uniform rationing* rule where all winning bids are awarded on a pro-rata basis, regardless of whether they are above or at the stop-out price; formally:

$$q_n = \frac{x_n(p_0)}{\max\{X(p_0), Q\}} Q. \tag{15}$$

This rule eliminates underpricing by incentivizing bidders to increase demand at the stop-out price if it is less than v. In turn, this makes the margin "large" and thus induces price competition, much like under the quantity discreteness scenario studied in Section 3 [see Kremer and Nyborg (2002) for details].

A potential problem with this approach is the implicit assumption that agents can absorb large quantities. When bidders face capacity constraints, they are severely restricted in their ability to capture larger quantities through increasing prices. As a result, underpricing can occur in equilibrium. To see this, suppose $\overline{Q} < Q$ and let $p_0 < v$ be given.

Suppose each bidder submits

$$x^*(p) = \begin{cases} \overline{Q} & \text{if } p \leq p_0. \\ 0 & \text{otherwise.} \end{cases} \tag{16}$$

In this case, the stop-out price is p_0. Under the uniform rationing scheme, bidders receive equal shares, and each bidder's payoff is $(v - p_0)Q/N$. Since no bidder can move the stop-out price on his own, (16) describes an equilibrium when $\overline{Q} < Q$.[15]

Note that (16) is not an equilibrium when $\overline{Q} \geq Q$, since an agent could then win all Q units by demanding $\overline{Q}$ at ε above p_0. Provided ε is sufficiently small, his payoff would be improved. This is also the case when bidders are capacity constrained and the auction size is uncertain. In particular, if there is a positive probability that the auction size falls below $\overline{Q}$, a bidder would do better using the deviation just described. He would get the entire auction with positive probability at only ε above p_0. In general, under uncertain supply, if there is a positive probability of underpricing, bidders have an incentive to shift up the lower parts of their demand functions in order to capture larger quantities. Provided the supply is sufficiently noisy, the upshot is that only $p_0 = v$ can be supported in equilibrium.[16]

In many cases, the assumption of a noisy supply cannot be supported. Hence, we examine next how one can modify the allocation rule in order to re-establish $p_0 = v$ as the unique equilibrium price when $\overline{Q} < Q_{\max}$.

Modified Majority Property (MMP) - when $X(p_0) > Q$, a bidder who demands (i) a fraction α of the aggregate demand at the stop-out price **or** (ii) αQ above the stop-out price gets at least $\alpha Q/2$. Formally, if $X(p_0) > Q$ then $x_n(p_0) \geq \alpha X(p_0)$ **or** $x_n^+(p_0) \geq \alpha Q$ implies that $q_n > \alpha Q/2$.

With capacity constraints, there is a clear advantage of allocation rules that satisfy this property. A bidder is rewarded for being aggressive even if he is unable to move the price. An example of an allocation rule that satisfies the Modified Majority Property (MMP) is the *hybrid allocation rule*, which is the average of the standard rule (3) and the uniform rationing rule (15). Formally, when $X(p_0) > Q$, we have:

$$q_n = \frac{1}{2}\frac{x_n(p_0)}{X(p_0)}\,Q + \frac{1}{2}\left\{x_n(p_0^+) + \frac{x_n(p_0) - x_n^+(p_0)}{X(p_0) - X^+(p_0)}\right\}. \tag{17}$$

Theorem 3 *Suppose the auction size is* Q *and that bidders are capacity constrained (*$\overline{Q} < Q$*). If the allocation rule satisfies the modified majority property and* $N > 2[(Q/\overline{Q}) + 1]$*, the only equilibrium stop-out price is* v*.*

To see the intuition behind this result, let us focus on the hybrid allocation rule. Like the standard rule, it discriminates in favor of inframarginal demand, but it does so only partially. Like the uniform rationing rule, the hybrid rule therefore makes available inframarginal supply to bids at the stop-out price. Thus the hybrid rule creates incentives for bidders to be aggressive above, as well as on, the margin. By shifting up the lower part of their demand schedules, bidders can capture significant increases in quantity by negligible increases in price. This fosters price competition and the upshot is that underpricing is eliminated. For example, (16) is not an equilibrium because a bidder can increase his allocation by $0.5\,(\overline{Q} - Q/N)$ without moving the stop-out price by demanding $\overline{Q}$ above p_0.

4.1 Discreteness

We close this section by discussing discreteness in admissible demand functions. This allows us to compare our results to the previous section. We consider both a price tick, h, and a quantity multiple, w. However, we will see that the quantity multiple plays little role in the analysis. Suppose the stop-out price is $t > 1$ ticks below v and consider a player who originally was allocated no more than Q/N. Let $\lambda > Q/N$ denote the amount he gets by demanding the maximal amount at a price which is one tick higher. A sufficient condition for a profitable deviation is:

$$\lambda (t-1) > \frac{Q}{N} t,$$

or, equivalently,

$$t > t^{**} \equiv \frac{\lambda}{\lambda - \frac{Q}{N}}.$$

Hence we conclude that an upper bound on equilibrium underpricing is that the number of ticks is bounded by the ratio of the amount an agent gets by deviating to the increase in his allocation.

If the allocation rule satisfies the Modified Majority Property, we have $\lambda \geq \frac{\overline{Q}}{(Q+\overline{Q})} \frac{Q}{2}$. As in Theorem 3, we assume that $N > 2[(Q/\overline{Q}) + 1]$ to guarantee that $\lambda > Q/N$. Thus since t^{**} is decreasing in λ, an upper bound on underpricing, measured in ticks, is:

$$t^{**}_{MMP} = \frac{\overline{Q}N}{\overline{Q}(N-2) - 2Q}, \tag{18}$$

where the MMP subscript refers to the allocation rule. Note that the assumption on N and $\overline{Q}$ above (and in Theorem 3) is equivalent to saying that $\overline{Q} > \frac{2Q}{N-2}$. So this also assures that the denominator in (18) is positive. As an example, suppose that $\overline{Q} = Q/3$, approximately the same as in US Treasury auctions. We have $\lambda \geq Q/8$ and $t^{**}_{MMP} = \frac{N}{N-8}$. Thus if $N > 16$, the largest equilibrium underpricing would be 1 tick below v.

Comparing t^{**}_{MMP} with the underpricing bound in the discretized standard uniform price auction in Theorem 2, we see that an advantage of allocation rules satisfying the Modified Majority Property (MMP) is that they lead to lower underpricing when the quantity multiple is "small." This confirms the intuition from Theorem 3. Also note that in case of underpricing, an allocation rule that satisfies MMP gives better incentives for agents to deviate. The standard allocation rule may give them only one additional quantity multiple while an allocation rule that satisfies MMP results in a bigger increase in their allocation. This may have an impact in cases where agents collude using explicit or implicit agreements. An allocation rule that satisfies MMP is more effective in deterring such collusion.

5 Supply Uncertainty and Uniqueness

In this section, we revert to the original assumptions of the basic model in Section 1. In particular, the allocation rule is the standard one, (3). The objective is to characterize the complete set of supply uncertainty robust equilibrium demand schedules, according to the definition below. No restrictions are imposed on the set of admissible demand schedules.

Definition 1 *The demand schedule $x(p)$ is* **supply uncertainty robust** *if for any distribution of supply with support in the interval $[0, Q_{\max}]$ it is equilibrium for each bidder to submit $x(p)$.*[17]

Equilibria that are supply uncertainty robust are important for several reasons. First, they can be played from auction to auction even though the underlying conditions that affect the distribution of supply may change. Second, the robust equilibria also handle private information on supply. To see this, if $x(p)$ is supply uncertainty robust, then if all bidders, except one, submit $x(p)$, it is optimal for the remaining bidder also to submit $x(p)$ irrespective of the distribution of supply. Hence, whatever the bidder's private information on supply may be, he cannot do better than to submit $x(p)$. A trivial example of a supply uncertainty robust equilibrium is that all bidders submit x with $x(v) = \overline{Q}$, for $\overline{Q} > \frac{Q_{\max}}{N-1}$. Our focus below will be on robust underpricing equilibria.

The robust demand schedules of Back and Zender (the nonlinear equilibria in Proposition 1) have been derived by requiring the first order condition of the maximization problem (6) to hold at every price on $[0, v)$. But in general, a robust demand schedule needs to satisfy the first order condition, if it is differentiable, only on the set of stop-out prices induced by that demand schedule. So it is not obvious that these equilibria are unique in being supply uncertainty robust. However, this is what we will proceed to show.

Denote by $s(x, Q)$ the stop-out price if all bidders submit the demand function $x(p)$ and the auctioned supply is Q. Additionally, define $S(x)$ to be the set of all possible stop-out prices under $x(p)$. That is,

$$S(x) = \{\, p \mid \exists\, Q \in (0, Q_{\max}] \text{ s.t. } p = s(x, Q)\,\}.$$

Proposition 2 *Suppose $\overline{Q} > \frac{Q_{\max}}{N}$ and $x(p)$ is a supply uncertainty robust equilibrium demand schedule which for some Q leads to a stop-out price below v. The set of possible stop-out prices, $S(x)$, has the form $[r, v)$, where $r \in [0, v)$. Furthermore, $x(p)$ is continuous and strictly decreasing on (r, v), and $x(v) = 0$.*

The proposition is proved in the appendix by a sequence of four lemmas. The first two lemmas establish that a robust underpricing equilibrium demand schedule is continuous at every possible stop-out price which is less than v. Intuitively, if the demand schedule had a discontinuity from below, there would be excess supply for some Q. This is incompatible with equilibrium if Q were known to occur, since the excess supply could be picked up costlessly by any bidder. If there were a discontinuity from above, there would be excess demand for some Q. But then some bidder could gain a significant increase in quantity from a negligible, if any, increase in the stop-out price.

The third lemma establishes that $x(v) = 0$ and $x(p) > 0$ for all $p < v$. The first statement implies that there will always be underpricing, even for small Q. If not, bidders would have an incentive to lower their demand functions so as to induce underpricing. If the second statement did not hold, then a bidder could capture a "large" increase in quantity by a "small" increase in price through increasing his demand at prices "close to" v. An implication of the first three lemmas is that there is an equilibrium stop-out price arbitrarily close to v.

The fourth lemma establishes that $x(p)$ is strictly decreasing on (r, v), where r is the smallest possible stop-out price (which occurs when $Q = Q_{\max}$). The intuition is that a "flat" provides in some circumstances an opportunity for a bidder to bring about a large decrease in the stop-out price at a cost of only a small reduction in his share. Thus there is a unique market clearing price for a given Q. Furthermore, in conjunction with the other lemmas, this implies that if p' is a possible stop-out price in a robust equilibrium, then any price between p' and v is also a possible stop-out price. Hence $S(x)$ has the form $[r, v)$.

With Proposition 2 in hand, it is relatively straightforward to prove the supply uncertainty robust uniqueness of Back and Zender's equilibria.

Theorem 4 (Uniqueness) *Suppose $\overline{Q} > \frac{Q_{\max}}{N}$ and $x(p)$ is a supply uncertainty robust equilibrium demand schedule which for some Q leads to a stop-out price below v. Then*

$$x(p) = \begin{cases} a(1 - p/v)^{\frac{1}{N-1}} & \text{if } p \geq \bar{p} \\ \overline{Q} & \text{otherwise,} \end{cases} \tag{19}$$

where $a > \frac{Q_{\max}}{N}$ and $\bar{p} = \left(1 - (\overline{Q}/a)^{N-1}\right) v$, or $x(p)$ is some trivial variation of this.[18]

It is noteworthy that while any monotonically decreasing demand function could, in principle, be submitted by a bidder, the robust demand schedules turn out to be strictly concave and differentiable. The concavity is interesting because it reflects strategic concerns rather than preferences as such. The intuition relates to the tradeoff faced by bidders between price and quantity. A bidder can increase the quantity awarded to him only by raising the stop-out price. And he can lower the stop-out price, only by lowering the quantity awarded to him. Thus, for a stop-out price below v to be supported in equilibrium, demand schedules must be such that a "small" increase in price will yield only a "small" gain in quantity. Likewise, a "small" reduction in the stop-out price must lead to a "large" loss in quantity. This is essentially a convexity condition on the residual supply, since the tradeoff is faced at a continuum of points when the auctioned supply can by anything, and this translates into a concavity condition on the demand schedules. Under the unique supply uncertainty robust demand schedules, underpricing can be reduced by adding bidders, and the stop-out price approaches v as the number of bidders approach infinity [see (11)].

We saw above that there may exist underpricing equilibria in the discrete uniform auction when there is both a tick size and a quantity multiple. Next we address whether any of these equilibria are robust to supply uncertainty.

Theorem 5 *Suppose there is a tick size, h, and a quantity multiple, w. Suppose also that A2 and A3 hold and consider distributions of $\tilde{Q}$ with realizations Q that are divisible by w. In any supply uncertainty robust equilibrium, the stop-out price is always v.*

The intuition derives from the property of supply uncertainty robust equilibria that strategies are *ex post* optimal. That is, once supply is known, a bidder will not want to change his strategy. But as discussed in Section 3.2, in a discrete setting when quantity is known bidders always want to load up demand at the stop-out price in order to get as big a share as possible of the rationed margin. Thus bidders will generally wish to change their strategies *ex post*. This is further illustration of the dramatic reduction in market power brought about by discreteness in bids.

6 Concluding Remarks

This paper has studied underpricing in uniform price auctions using the theoretical framework of Wilson (1979) and Back and Zender (1993). We ask questions that we believe are of empirical or normative importance: Does equilibrium underpricing survive simple modifications which make the model more realistic? Can we modify the auction so as to improve its performance? Is there any reasonable way to pick among the plethora of equilibria for someone possessing individual bidding data and who wishes to test the underpricing and market power hypothesis?

In their analysis, Wilson and Back and Zender place few restrictions on admissible demand functions. Yet in practice, bidders normally do not specify explicit demand functions, but rather they submit finite collections of price-quantity pairs as bids. Moreover, there is a minimal price tick and a minimal quantity multiple. We have shown that this has profound impact on the set of equilibria; underpricing can be reduced or even eliminated. The reason is that discreteness makes the marginal residual supply "large". In turn, this creates price competition for the marginal units and thus reduces underpricing. The effect is robust to supply uncertainty and to players being capacity constrained.

We have also shown that a simple change to the allocation rule can go a long way in reducing underpricing. The change involves making inframarginal supply available to marginal bids. This tends to push the stop-out price up because it stimulates aggressive bidding on the margin, along similar lines as in the discrete case. The choice of allocation rule depends on whether bidders face capacity constraints, that is, whether a single bidder can absorb all the supply. When there are no capacity constraints, an example is the *uniform rationing* rule. This gives each bidder a prorated portion of the supply, irrespective of whether his bids are above or at the stop-out price. When there are capacity constraints, the design takes into account the fact that a single bidder may be unable to move the stop-out price. An example is that a bidder gets half of what he would get under the standard allocation rule and half of what he would get under the uniform rationing rule. This may shed light on the common practice in bookbuilding in IPO's to encourage a build-up of excess demand and using an allocation rule which gives only partial priority to infra-marginal demand.

Testing the Wilson/Back and Zender model is not simple. There are many underpricing equilibria to choose from. Furthermore, the characteristics of the different equilibria can be very different. For example, Proposition 1 describes both linear and concave equilibria and this is just the tip of the iceberg. However, we have shown, without imposing any restriction on the set of admissible demand functions, that there is a unique class of symmetric equilibria which are robust to supply uncertainty and private information on supply. This is important because supply uncertainty is often present in practice, for example due to demand from non-competitive bidders (Back and Zender, 1993).

While this paper has focused on a setting of complete information, in practice players in treasury and other multiunit auctions may well have private information, as suggested by Cammack (1991). Recent empirical evidence by Nyborg, Rydqvist, and Sundaresan (2002), Keloharju, Nyborg, and Rydqvist (2003), and Bjønnes (2001) provide some guidance as to how this is likely to affect bidder behavior and auction performance. Theoretical advances have been made by, for example, Kyle (1989) and Wang and Zender (2002). However, from a theoretical perspective, the findings of this paper suggest that the results one will find may well depend on whether bidders are modelled as submitting smooth or discrete demand functions, something which would be an important issue to address in future research.

Appendix

Proof of Lemma 1

Since $(N-1)x(v) < Q$, bidder 1 clearly can and will choose his demand schedule, denoted by y, such that $p_0 < v$ and therefore also such that supply is exhausted at p_0. Furthermore, the residual supply curve facing bidder 1, $R_{-1}(p) = \max[Q-(N-1)x(p), 0]$, is continuous and strictly increasing when it is positive. As a result, for any p_0 that bidder 1 can generate through his choice of y, bidder 1's award is $Q-(N-1)x(p_0)$, since all demand at prices above p_0 is awarded in full according to (3). Hence, bidder 1 can do no better than picking y such that $y(p_0)+Nx(p_0) = Q$. Since bidder 1's payoff is not affected by his demand at prices above or below p_0, he may as well submit a differentiable and strictly decreasing function. □

Proof of Proposition 1

Linear Equilibria:

Substituting (8) into (7), the first order condition can be written

$$(N-1)(v-p_0)x'(p_0) + \frac{Q}{N} = 0. \tag{20}$$

We claim that there is a linear demand function

$$x(p) = \alpha + \beta p \tag{21}$$

which is submitted by each bidder in equilibrium. This will now be verified. Differentiating (21) and substituting into the first order condition (20) leads to $\beta = -\frac{Q}{(v-p_0)N(N-1)}$. Let $\gamma(p_0)$ be such that $p_0 = \gamma(p_0)v$. We will turn this relation around, showing that the choice of γ will uniquely determine p_0. Writing γ as a parameter between 0 and 1,

$$\beta = -\frac{Q}{(1-\gamma)vN(N-1)}. \tag{22}$$

It remains to pick α such that the market clearing condition is satisfied at $p_0 = \gamma v$. Thus, α must satisfy $N\left(\alpha - \frac{\gamma v Q}{(1-\gamma)vN(N-1)}\right) = Q$. Hence,

$$\alpha = \frac{Q}{N} + \frac{\gamma Q}{(1-\gamma)N(N-1)}. \tag{23}$$

Substituting the equations for β and α back into (21), the general expression for linear equilibrium demand schedules is obtained:[19]

$$x(p) = \frac{Q}{N} + \frac{\gamma Q}{(1-\gamma)N(N-1)} - \frac{pQ}{(1-\gamma)vN(N-1)}.$$

By construction, the stop-out price is γv, irrespective of N. Since γ can be any number between 0 and 1, this also implies that any stop-out price between 0 and v can be obtained in equilibrium. The second order condition is satisfied since the demand function is linear.

Nonlinear Equilibria

Substituting (8) into (7), the first order condition can also be written[20]

$$(N-1)(v-p_0)x'(p_0) + x(p_0) = 0. \tag{24}$$

By requiring (24) to hold for all p, (24) becomes a differential equation, the solutions to which will yield a new class of equilibrium demand schedules. The general solution to this differential equation is

$$x(p) = c(v-p)^{\frac{1}{N-1}}, \tag{25}$$

where c is a positive constant. Note that (25) also satisfies the second order condition. Letting $a = cv^{\frac{1}{N-1}}$, equation (25) can be rewritten as

$$x(p) = a(1-p/v)^{\frac{1}{N-1}}. \tag{26}$$

Since $a = x(0)$, it follows that a can be any number not smaller than Q/N (see footnote 6). The formula for the stop-out price, (11), follows by substituting $Nx(p_0) = Q$ into (26). Thus, the stop-out price can be any number between 0 and v. □

Proof of Theorem 1

Consider first the case that bidders can submit an arbitrary finite number of bids (the rules of the auction do not specify a maximum number). Suppose by contradiction that there is an equilibrium with stop-out prices below v and let $\underline{p} < v$ denote the lowest equilibrium stop-out price. The idea behind the proof is to construct a deviation for a bidder in which he demands the maximal amount

$\overline{Q}$ at a price $\underline{p}+\varepsilon$ and keeps his original strategy at higher prices. This deviation has an effect only in cases in which previously prices were between $\underline{p}$ and $\underline{p}+\varepsilon$. If we show that this bidder increases his allocation by an amount bounded away from zero for arbitrarily small $\varepsilon > 0$, we would get a contradiction. For small enough ε this would be a profitable deviation as price effects would be negligible.

We begin by noting that since there is a finite number of bids, after observing the bids made we can find an $\varepsilon^* > 0$ so that there are no bids between $\underline{p}$ and $\underline{p}+\varepsilon^*$. This implies that there exists a set of constants $\{c_j^*\}_{j=1}^N$ so that $x_j(p) = c_j^*$ for $p \in (\underline{p}, \underline{p}+\varepsilon^*)$. We also conclude that $\sum c_j^* < Q_{\max}$ and the price $\underline{p}$ occurs with positive probability. Hence, if some bidder j demands less than $\overline{Q}$ at $\underline{p}$ he can do better by adding the bid $(\underline{p}, \overline{Q} - c_j^*)$ which increases his demand to $\overline{Q}$ at $\underline{p}$. Thus, all bidders must demand $\overline{Q}$ at $\underline{p}$. Since $(N-1)\overline{Q} > Q_{\max}$, this implies that at least two bidders have bids at $\underline{p}$ and therefore that bids at $\underline{p}$ will be rationed in the event that the stop-out price is $\underline{p}$.

Using the above conclusions we focus on an agent who originally has a bid at $\underline{p}$ and therefore demands less than $\overline{Q}$ at any higher price; that is, a bidder i for whom $c_i^* < \overline{Q}$. Pick an $\varepsilon < \varepsilon^*$ and consider the deviation of moving the original bid for $\overline{Q} - c_i^*$ at $\underline{p}$ up to $\underline{p}+\varepsilon$. We argue that this bidder increases his allocation by an amount bounded away from zero. To see this, condition on the realized supply $\tilde{Q}$ and let p^{old} and p^{new} denote the old and new stop-out prices, respectively. There are two cases to consider:

- $p^{old} = \underline{p}$ and $p^{new} = \underline{p}+\varepsilon$: Originally the aggregate demand at $\underline{p}$ was $X(\underline{p}) = N\overline{Q} > \tilde{Q}$, and at a slightly higher price it was $\sum_j c_j^* < \tilde{Q}$. Hence, by (3) bidder i got

$$q_i^* = c_i^* + R^* \alpha_i^* \tag{27}$$

where

$$R^* = \tilde{Q} - \sum_j c_j^* \qquad \text{and} \qquad \alpha_i^* = \frac{x_i(\underline{p}) - c_i^*}{X(\underline{p}) - \sum_j c_j^*}. \tag{28}$$

As a result of the deviation, the stop-out price moves to $\underline{p}+\varepsilon$ and the bidder gets an allocation of $c_i^* + R^*$. Since $\alpha_i^* < 1$ (since at least two bidders submit bids at $\underline{p}$), this is more than q_i^*.

- $p^{old} = p^{new} = \underline{p}$: As in the previous case, bidder i originally got $q_i^* < \overline{Q}$ as given by (27), whereas as a result of the deviation he gets the maximal possible amount $\overline{Q}$.

Since ε is arbitrarily small and since the new allocation is independent of ε and strictly larger than the old allocation, this establishes that if $\underline{p} < v$ and bidders can submit any number of bids, some bidder has a profitable deviation and therefore this is inconsistent with equilibrium.

Consider next the case that the number of bids a bidder can submit is uniformly bounded (the rules of the auction specify a maximum number of bids for each bidder). Define $\underline{p} < v$ as before. If at least two bidders have bids at $\underline{p}$, then along the same lines as above, either one of them can improve his payoff by moving his bid at $\underline{p}$ up slightly (so as to avoid rationing). Suppose next that at most one bidder has a bid at $\underline{p}$. Since $(N-1)\overline{Q} > Q_{\max}$, this implies that at least two bidders demand less than $\overline{Q}$ at $\underline{p}$. By assumption, one of them does not have any bids at $\underline{p}$. If the stop-out price is $\underline{p}$ with probability one, then this bidder can increase his allocation with at most a negligible increase in price by dropping all his bids and replacing them with one bid for $\overline{Q}$ slightly above $\underline{p}$. Therefore the probability that the stop-out price is $\underline{p}$ must be less than one. Now order all bids from lowest to highest and let p_1 be the first price level above $\underline{p}$ where there is a bid and which has a positive probability of being a stop-out price. Then if at least two bidders have bids at p_1, along the same lines as above, either one of them can improve his payoff by moving up his bid slightly (so as to avoid rationing). If only one bidder has a bid at p_1, he could do better by lowering it. These profitable deviations establish that $\underline{p} < v$ is inconsistent with equilibrium.

To finalize the proof, note that there is an equilibrium where the stop-out price is v almost surely since it is equilibrium for all bidders to bid $\overline{Q}$ at a price of v, since $(N-1)\overline{Q} > Q_{\max}$. □

Proof of Theorem 2

It is obvious that v and $v-h$ can be supported as stop-out prices in equilibrium. Therefore t^* must be bounded below by 1. Consider therefore $p_0 = v - t_0 h$, where $t_0 \in \{2, \ldots, v/h\}$. We do not consider $t_0 > v/h$ since the reservation price is 0. We first claim that if p_0 can be supported in equilibrium it must be that (i) $x_i(p_0) = \overline{Q}$ and (ii) $X(p_0+h) = Q - w$. These claims follow immediately when $w = Q$. They are also true otherwise; since if not, some bidder can increase his allocation without changing the stop-out price. In particular, if (i) does not hold for bidder i, he can rearrange his bids into two bids, one above p_0 and one at p_0. By having $x_i(p_0) = \overline{Q}$, he maximizes his allocation. If (ii) does not hold, some bidder can increase his allocation by demanding w more units above p_0.

Observe now that a bidder has the smallest incentive to deviate if (iii) $X(v) = \overline{Q} - w$. Therefore, p_0 can be supported in equilibrium if and only if it can be supported when (iii) holds; hence, we assume (iii). In this case, the best deviation for any bidder i, who initially has demand of less than $\overline{Q}$ at $p_0 + h$, is to submit a single bid for $\overline{Q}$ units at $p_0 + h$. This deviation results in the stop-out price $p_0 + h$. Letting q_i denote the original allocation of agent i and $\hat{q}_i$ his allocation when following the above deviation, we conclude that p_0 is an equilibrium stop-out price if and only if for all i with $x_i(p_0+h) < \overline{Q}$

$$q_i\left(v - p_0\right) \geq \hat{q}_i\left(v - p_0 - h\right).$$

Dividing both sides by h and rearranging tells us that $p_0 = t_0 h$ can be supported in equilibrium if and only if

$$t_0 \leq \frac{\hat{q}_i}{\hat{q}_i - q_i}. \tag{29}$$

To express $\frac{\hat{q}_i}{\hat{q}_i - q_i}$ in terms of the parameters of the model, first note that as a result of the above deviation, bidder i secured the last unit, before it was being rationed. That is, if we let y_i denote i's original demand at $p_0 + h$ then $\hat{q}_i = y_i + w$ and $q_i = y_i + \alpha\left(y_i\right) w$, where $\alpha(y) = \frac{\overline{Q} - y}{N\overline{Q} - (Q - w)}$. Now define $f(y) = \frac{y + w}{[1 - \alpha(y)]w}$. Then $f(y_i) = \frac{\hat{q}_i}{\hat{q}_i - q_i}$ and $f(y)$ is increasing. Hence, by (29), $p_0 = v - t_0 h$

is an equilibrium stop-out price if and only if $t_0 \leq f(y_{\min})$, where $y_{\min} = \min\{y_i\}_{i=1}^N$. It follows that p_0 can be supported in equilibrium if and only if $t_0 \leq f(y^*)$, where $y^* = \max\{\min_S\{y_i\}\}$ and S is the set of demand profiles satisfying (i) and (ii) above. Thus we can conclude that for $t_0 \in \{0, 1 \ldots, v/h\}$, $p_0 = v - t_0 h$ can be supported in equilibrium if and only if $t_0 \leq t^* \equiv \max\{1, f(y^*)\}$.

To establish bounds on t^*, focus on y^* and note that since the largest of the smallest y_i happens when the bidders are as equal as possible, we have $y^* = I\left[\frac{Q-w}{Nw}\right]w$, where $I[z]$ is defined to be the greatest integer smaller than or equal to z. Thus $(Q - Nw)/N \leq y^* \leq (Q - w)/N$. Since $f'(y) > 0$, an upper bound for $f(y^*)$ is $f((Q-w)/N) = \frac{Q}{(N-1)w} + 1$, and a lower bound is $f((Q - Nw)/N) \geq \frac{Q}{(N-1)w}$. This establishes the theorem. □

Proof of Theorem 3

Suppose to the contrary that $p_0 < \bar{v}$. We start by noting that there is a bidder, call him i, who is awarded at most Q/N. Consider the deviation where for some $\varepsilon > 0$, he demands $\overline{Q}$ at $p_0 + \varepsilon$ and zero at higher prices. Let p^{new} denote the new stop-out price. There are two cases to consider:

- $p^{new} = p_0$: In this case, bidder i demands $\overline{Q}$ above the stop-out price and, by the definition of the *Modified Majority Property*, is awarded at least $\overline{Q}/2$. Since $N > 2(1/\overline{Q} + 1) > 2/\overline{Q}$, for sufficiently small ε this is a profitable deviation.

- $p^{new} = p_0 + \varepsilon$: In this case, bidder i demands $\overline{Q}$ at the new stop-out price and the aggregate demand at this price is at most $\overline{Q} + Q$. By the definition of the *Modified Majority Property*, he is awarded at least $\overline{Q} \,/\, \left[2(\overline{Q} + Q)\right]$. Since $N > 2(Q/\overline{Q} + 1)$, for sufficiently small ε this is a profitable deviation. □

Proof of Proposition 2

The proposition will be proved by a sequence of lemmas. But first, define

$$S^o(x) = \{p | \exists Q \in (0, Q_{\max}) \text{ s.t. } p = s(x, Q)\}.$$

Lemma 2 *Suppose $p_0 \in S^o(x)$ and $p_0 < v$. $x(p)$ is continuous from below at p_0.*

Proof: Let $Q_0 = Nx(p_0)$. Suppose that $Q_0 < \lim_{p \to p_0^-} Nx(p) \equiv Q_0^-$. Then there is some $Q^* \in (Q_0, Q_0^-)$ for which there is excess supply. Let $\varepsilon = \min\left\{\overline{Q} - \frac{Q_0}{N}, Q^* - Q_0\right\}$. If Q^* is known to obtain, bidder 1's payoff under the equilibrium strategy $x(p)$ is $\frac{Q_0}{N}(v - p_0)$. But since $\overline{Q} > Q_{\max}/N$, he could increase this to $\left(\frac{Q_0}{N} + \varepsilon\right)(v - p_0)$ by demanding $x(p_0) + \varepsilon$ at p_0. This contradicts that $Q_0 < Q_0^-$. Hence $Q_0 = Q_0^-$ (since $x(p)$ is decreasing), which establishes that at p_0, $x(p)$ is continuous from below. □

Lemma 3 *Suppose $p_0 \in S(x)$ and $p_0 < v$. $x(p)$ is continuous from above at p_0.*

Proof: Suppose $Q_0^+ \equiv \lim_{p \to p_0^+} Nx(p) < Q_0 \equiv Nx(p_0)$. Then there is $Q^* \in (Q_0^+, Q_0)$ for which there is excess demand. Let $\delta = \min[Q^* - Q_0^+, \overline{Q} - Q_0^+/N]$. Note that $\delta > (Q^* - Q_0^+)/N$ since $Q^* < Q_0 \leq N\overline{Q}$. If Q^* is known to obtain, bidder 1's payoff under the proposed equilibrium strategy $x(p)$ is $\pi_1 = \left(\frac{Q_0^+}{N} + \frac{Q^* - Q_0^+}{N}\right)(v - p_0)$. Consider that the bidder deviates by submitting, for some $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$,

$$\hat{x}_1(p) = \begin{cases} x(p) & \text{if } p > p_0 + \varepsilon_1 \\ x(p) + \delta(1 - \varepsilon_2) & \text{if } p \leq p_0 + \varepsilon_1. \end{cases}$$

This gives him a payoff of $\hat{\pi}_1 = (v - p_0)\left(\frac{Q_0^+}{N} + \delta(1 - \varepsilon_2)\right)$, which exceeds π_1 for sufficiently small ε_2. This contradicts that $Q_0^+ < Q_0$, which establishes that $x(p)$ must be continuous from above at p_0 (since $x(p)$ is decreasing). □

These two lemmas show that for every supply uncertainty robust equilibrium demand schedule under which there are $p_0 < v$, $x(p)$ is continuous at every $p_0 \in S^o(x)$.

Lemma 4 *$x(v) = 0$; and for $p < v$, $x(p) > 0$.*

Proof: We will first show that $x(v) = 0$. Suppose to the contrary that $x(v) = Q_v/N > 0$. Note that $Q_v < Q_{\max}$ since, by assumption, there is some Q for which $p_0 < v$. Let Q_v be known to obtain. Then each bidder's payoff is 0. Let $p^* = (\inf\{p|(N-1)x(p) < Q_v\} + v)/2$. When the other bidders submit x and Q_v is known to obtain, a bidder can guarantee himself a positive payoff by submitting

$$z(p) = \begin{cases} x(p) & \text{if } p \leq p_0 \\ 0 & \text{if } p > p_0. \end{cases}$$

Hence the bidder is better off by submitting $z(p)$, which contradicts that $x(v) > 0$. Hence $x(v) = 0$.

Next, we will show that for all $p < v$, $x(p) > 0$. Suppose to the contrary that this is not the case. Let $p^* = \min\{p|x(p) = 0\}$, which exists by continuity. By continuity, there is $p_0 < p^*$ and $Q_0 < Q_{\max}$ such that $Q_0 = Nx(p_0)$. If Q_0 obtains, each bidder's payoff under the proposed equilibrium is $\frac{Q_0}{N}(v - p_0)$. If Q_0 is known to obtain and bidder 1 deviates by demanding Q_0 at a price of p^*, his payoff is $Q_0(v - p^*)$. This is larger than what he gets in the proposed equilibrium if and only if

$$v - p^* > \frac{v - p_0}{N}. \tag{30}$$

Since $N \geq 2$, continuity assures that there is p_0 such that (30) holds. This contradicts the assumption that $p^* < v$, thereby establishing the lemma. □

By Lemma 4, $\sup S(x) = v$. Let $r = s(x, Q_{\max}) = \min S(x)$. Continuity implies that $Nx(r) = Q_{\max}$.

Lemma 5 *$x(p)$ is strictly decreasing on (r, v).*

Proof: Suppose $x(p)$ is not strictly decreasing. Then there is some open interval $(b^o, c^o) \subset (r, v)$ on which $x(p)$ is constant. Let $Nx(p) = Q^*$ for $p \in (b^o, c^o)$. Let $c = \sup\{p|Nx(p) = Q^*\}$ and let $b = \inf\{p|Nx(p) = Q^*\}$. By continuity, $Nx(p) = Q^*$ on the closed interval $[b, c]$. If Q^* occurs, according (2), the stop-out price is c.

Consider the case that Q^* is known to obtain. Some bidder can improve his payoff by changing his demand schedule to

$$w(p) = \begin{cases} 0 & \text{if } p > b \\ Q^*/N & \text{if } p = b \\ x(p) & \text{otherwise.} \end{cases}$$

This provides the bidder with the same award as under $x(p)$, but the stop-out price is now b rather than $c > b$ as under $x(p)$. Hence the bidder is better off, implying that $x(p)$ must be strictly decreasing on (r, v). □

The above lemmas imply that $S(x)$ has the form $[r, v)$ and that $x(p)$ is continuous and strictly decreasing on (r, v). It is continuous from below at v and from above at r. This completes the proof of Proposition 2.

Proof of Theorem 4

By Proposition 2, there is $r \in [0, v)$ such that $S(x) = [r, v)$. Since $x(p)$ is monotonically decreasing on $[r, v)$, it is differentiable almost everywhere on $[r, v)$ [Titchmarsh (1986), p. 350 and p. 358]. Hence, $x(p)$ must satisfy the first order condition (24) almost everywhere on (r, v). In particular, for any p^* at which $x(p)$ is differentiable, there is a number $a(p^*) > 0$ such that $x(p^*) = a(p^*)(1 - p/v)^{\frac{1}{N-1}}$. Continuity implies that $a(p^*)$ is independent of p^* and can be denoted simply by a. For reasons discussed in footnote 6, equilibrium requires that $x(0) \geq \frac{Q_{\max}}{N}$. Since also $x(0) \leq \overline{Q}$, this establishes that $x(p)$ must be given by (19), or some trivial variation thereof [i.e. different only on prices outside $[r, v)$, where $r = \left(1 - (Q_{\max}/Na)^{N-1}\right) v$]. □

Proof of Theorem 5

Suppose by contradiction that $\mathbf{x}$ is a supply uncertainty robust underpricing equilibrium profile. Thus $\mathbf{x}$ must satisfy *ex post* optimality. But if the supply, Q, is known then the stop-out price, $p_0(Q)$, is also known. Therefore, if $p_0(Q) < v$, every bidder must optimally choose $x_n(p_0(Q)) = \overline{Q}$. Thus there can be at most one equilibrium stop-out price below v. Denote this by p_0^*.

The proof of Theorem 2 shows that when the supply is known, the equilibrium residual supply at p_0^* must be w. Therefore, the stop-out price is p_0^* only when $Q = Q_{\max}$ and v otherwise. In particular, the aggregate demand must satisfy

$$X(p) = \begin{cases} N\overline{Q} & \text{if} \quad 0 \leq p \leq p_0^* \\ Q_{\max} - w & \text{if} \quad p_0^* < p \leq v \\ 0 & \text{if} \quad p > v. \end{cases}$$

But then, if $Q = Q_{\max} - w$, the stop-out price is v and bidders earn zero profits. Hence some bidder could do better by lowering his bids to move the stop-out price below v. This is a contradiction and thereby establishes the theorem. □

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Notes

[1]Since September 1998, all US Treasury auctions have been conducted on a uniform basis. In the UK, inflation linked gilts are auctioned this way.

[2]Monopsonistic market power also plays an important role in Kyle (1989) and Wang and Zender (2002), where players are risk averse, and Ausubel and Cramton (1998), where players are capacity constrained.

[3]The quotation is taken from Larry Summers' foreword to Malvey and Archibald (1998).

[4] In US and UK Treasury auctions, for example, there are two types of bidders. Non-competitive bidders submit bids that specify quantities only, while competitive bidders submit price-quantity pairs. Since non-competitive bids are allocated first, they reduce the amount available to competitive bidders, who may therefore view the amount available to them as random (Back and Zender, 1993).

[5]The common value v could represent the expectation of some uncertain value. To motivate the use of the auction, it could be assumed that the seller does not know v and wishes to distribute the goods to several players. The US Treasury has dispersed awards as an auction objective (see the *Joint Report*). Wilson (1979) and Back and Zender (1993) also consider private information on v. However, in Back and Zender, it is not used in equilibrium. In Wilson (1979), the equilibrium with private information is fully revealing but underpricing arises for the same reason as in his equilibrium without private information.

[6] $x(0) < Q/N$ is not consistent with equilibrium. For in this case, the stop-out price would be zero and any bidder could do better by bidding $\hat{x}$ with $\hat{x}(0) = \overline{Q}$.

[7]The maximization problem can also be written $\max_{p_0} v[Q - (N-1)x(p_0)] - p_0[Q - (N-1)x(p_0)]$. The first term can be interpreted as the bidder's total revenue and the second term as the bidder's total cost. Viewed in this way, the first order condition is the standard equalization of marginal revenue and marginal cost. The problem can also be written $\max_\pi \pi R(\pi)$, where $\pi = v - p_0$ is underpricing and $R(\pi)$ is residual supply as a function of underpricing. The first order condition of this says that at the optimal underpricing, the elasticity of the residual supply, with respect to the underpricing, is 1.

[8]See also Wang and Zender (2002). There are numerous other equilibria than those in Proposition 1.

For example, any demand function $z(p)$ satisfying the following three conditions will do: (i) $z(p_0) = Q/N$, and $z(p)$ is (ii) less steep than (9) for prices above p_0, and (iii) steeper than (9) for prices below p_0.

[9]Wilson's (1979) equilibrium is the special case of the linear equilibria that $\gamma = .5$.

[10]See *Official Operations in the Gilt-Edged Market: Operational Notice*, by the UK Debt Management Office, November 2001.

[11]The contrast between Proposition 1 and Theorem 1 can also be understood in a more conventional way. In the underpricing equilibria, a bidder can only increase the quantity he gets by increasing the price by so much that his profits fall. There is an element of quantity competition here, and this creates positive profits along similar lines as in the Cournot oligopoly model [see Klemperer and Meyer (1989) for further discussion]. The no-underpricing result in Theorem 1 bears obvious resemblance to the result that Bertrand competition eliminates these profits.

[12]The case of only a quantity multiple is the limiting case as h goes to zero of Theorem 2. When there is a price tick but no quantity multiple, agents can increase their allocations without changing the stop-out price by demanding a fraction of the marginal units, $R^+(p_0)$, one tick above the stop-out price.

[13]The proof implies that an equilibrium with a stop-out price $p_0 = v - t_0 h$ for $t_0 \leq \min\{t^*, v/h\}$ is given by

$$x_n(p) = \begin{cases} \overline{Q} & \text{if } p \leq p_0 \\ y^* + w1_{[n>N-k]} & \text{if } p \in (p_0, v] \\ 0 & \text{if } p > v. \end{cases} \tag{31}$$

where $k = (Q - w - Ny^*)/w$, $y^* = I\left[\frac{Q-w}{Nw}\right] w$, and $I[z]$ is defined to be the greatest integer smaller than or equal to z. Note that $y^* + w < \overline{Q}$ since $(N-1)\overline{Q} > Q$.

[14]In Europe a price tick is more common while in the US a yield tick is used.

[15]This underpricing equilibrium example has been pointed out by Ulf Axelson. Note also that this particular equilibrium does not work under the standard allocation rule. A bidder could then increase his allocation without increasing the stop-out price

[16]A sufficient condition is that the support of the distribution of the supply is $[0, Q_{\max}]$. A proof is

available from the authors upon request.

[17]Note that this definition covers both continuous and discrete distributions.

[18]Let $r = \left(1 - (Q_{\max}/Na)^{N-1}\right) v$ and $z(p)$ be a monotonically decreasing function with $z(p) \geq \text{RHS}(19)$. A trivial variation of (19) would be (note that $\bar{p} < r$)

$$x(p) = \begin{cases} a(1-p/v)^{\frac{1}{N-1}} & \text{if } p \geq r \\ z(p) & \text{otherwise.} \end{cases}$$

[19]Given that the other $N-1$ bidders submit the demand function (9), bidder 1 maximizes his payoff by also using (9). There are other demand functions that would be equally good. But if bidder 1 did not submit (9), the other bidders would not necessarily be maximizing expected payoff by employing (9).

[20]Equation (24) differs from (20) in that Q/N has been substituted by $x(p_0)$.