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The relationship between the allocation of goods and a seller's revenue

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Abstract

We examine a seller auctioning off a set of objects to a large number of bidders, and restrict attention to auctions that are self-enforcing in that bidders do not want to walk away from the mechanism after they see the price that they must pay. We show that the seller's ability to extract the full possible revenue depends on whether the efficient allocation is concentrated among a few bidders or is dispersed so that a non-negligible fraction of bidders obtain objects. If it is concentrated then *any* sequence of mechanisms that achieve the efficient allocation (and there are many) asymptotically extracts the full surplus, and so it is in the seller's interest to efficiently allocate objects. In contrast, when the efficient allocation is dispersed then *no* sequence of mechanisms asymptotically extracts the full surplus. Moreover, in the dispersed case the seller may benefit by inefficiently bundling objects for sale.

Keywords: Mechanism design; Asymptotic analysis; Allocations

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1. Introduction

Consider a monopolist facing a large number of potential buyers, who may differ in their valuations for the goods for sale and/or in the information that they hold regarding the quality of the goods. The central question that we ask is whether the high degree of competition among the large number of buyers enables the seller to extract the full possible surplus from sale of the objects, and whether the seller has an incentive to allocate the objects in an efficient manner.

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It is well known, that with a limited number of bidders, a seller may gain from inefficiently allocating goods, for instance by setting a reserve price; and that bidders enjoy some rents for their private information (see Myerson (1981), Riley and Samuelson (1981), and Milgrom and Weber (1982), among others). It is also known, however, that as the number of bidders grows a seller of a single object can extract the full possible surplus from the sale of the object; and moreover, this is true regardless of whether the valuations are private or common. Any mechanism that efficiently allocates the object leads to the same full revenue (see Bali and Jackson (2002)). Thus, a seller of a single object to a large number of bidders has an incentive to efficiently allocate the object, extracts the full revenue, and is indifferent between a large number of auction formats. Here we examine whether the same is true when a seller is selling more than one object, and find that the answer depends on the structure of the efficient allocation of the objects.

In order to study this question, we impose a constraint on the auction formats that the seller may choose. In particular, Crémer and McLean, 1988 (see also McAfee and Reny (1992)) have shown that when there is even a minimal amount of correlation among buyers' types, then a seller can extract full rents through an elaborate payment scheme that depends in careful ways on the underlying distribution of types, and this is true even with small numbers of agents. Thus, there exist auction formats that implement an efficient allocation and extract the entire surplus. However, such mechanisms are extreme and are not used in practice, and might be thought as a theoretical benchmark rather than a practical device. There are several reasons why these might not be practical, and an important one is that agents are sometimes charged prices that they know to exceed the value of the good they obtain. Upon observing their own allocation and payment, agents often prefer to walk away, even without knowing anything about the information of the other agents. This means that the seller would face the unpleasant task of forcing the mechanism outcome on its participants.

The condition we impose is that a mechanism be *self-enforcing*, which effectively means that this never happens. The condition requires that an agent does not wish to leave after he or she observes his or her allocation and price—but not necessarily anything about the information of other agents. This is a condition that is slightly stronger than an interim individual rationality constraint, but substantially weaker than an ex post constraint. Under this condition, it is still possible that agents eventually regret having purchased an object, say in the case of common values. Nevertheless, this condition is strong enough to rule out the Crémer and McLean types of constructions, and is still satisfied by most standard auction formats that one can think of.[1]

With a restriction to self-enforcing mechanisms, our results may be summarized as follows. If the efficient allocation is concentrated in the hands of a few agents, then any mechanism that achieves the efficient allocation also extract the entire surplus. Moreover, such a mechanism always exists. Thus, with concentrated allocations, the seller has an incentive to efficiently allocate the objects for sale. In contrast, if the efficient allocation is dispersed among a non-negligible number of bidders, then *no* mechanism can extract the entire surplus. Moreover, in this case there is often tension between efficiency and revenue maximization even in the limit. In some situations a mechanism that inefficiently bundles objects for sale may lead to higher revenue than any mechanism that leads to an

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[1] An exception is an all-pay auction, where agents may be unhappy when they pay but do not get an object.

efficient allocation. Thus, with a large number of objects, a seller may have an incentive to inefficiently allocate objects, even when there are arbitrarily many potential buyers.

Finally, we note that for the case of concentrated allocations, we can pinpoint how fast the revenue of the seller grows with the number of buyers.

2. The setting

2.1. Economies

A sequence of economies is indexed by n, the number of agents in the economy. A non-random quantity Q^n of a good is to be sold in economy n. It may be fully divisible or may come in indivisible units.

2.2. Information

Information is described by a framework introduced by Milgrom (1981), as described in detail further. This information structure is particularly suited to the discussion of growing sequences of economies, all based on the same underlying information structure. We use upper case letters to denote random variables and lower case letters to denote realizations. We use f to denote a density or conditional density of a random variable and F to denote a distribution. In some cases when it may be unclear to which random variables we refer, we use subscripts such as $f_X(\cdot)$, while when it is clear we will omit the subscripts.

Each agent $i \in \{1, \dots, n\}$ in economy n observes a private signal S_i that takes on values in $[0, 1]$. There is also an underlying random variable X taking on values in $[0, 1]$. The S_i's are independently and identically distributed conditional on X. This conditional distribution of S_i given X is described by the density function $f(s_i|x)$. We assume that the unconditional (marginal) density of each S_i, $f(s_i)$, is positive for all s_i.

Note that we have not assumed any particular relationship between the signals and the underlying state X. For some of the later results, where we need to have some handle on the structure of equilibrium, we will assume affiliation. However, for some of the main results, this is not necessary. We also remark that we have assumed continuous distributions and identical distributions for the S_i's (conditional on X). This simplifies exposition, but is not necessary for the results. As will become clear in the proofs, what is necessary is that as the economy becomes larger, any given agent will face some competitive pressure from other agents who hold similar information.

Let S denote the vector of signals $S_1, \dots, S_n$ and let S_{-i} denote the vector of signals omitting S_i. Let $Y(k)$ denote the k-th order statistic of the signals S and let $Y_{-i}(k)$ denote the k-th order statistic of the signals S_{-i}.

We assume that:

Assumption A(I). $f(x|s_i)$ is continuous (in (x, s_i)), positive for all (x, s_i), and has a partial derivative with respect to s_i that is bounded above and below regardless of x.[2]

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[2] All that we require is that there exist $\alpha > 0$ such that for almost every x $|f(x|s_i) - f(x|s'_i)| < \alpha|s_i - s'_i| f(x|s'_i)$, which is implied by these assumptions. In fact, for most of our results it is sufficient to assume only continuity.

The important implication of this condition is that two nearby signals provide similar information about the realization of X.

2.3. Preferences

Agent i's valuation for the good is described by $v : [0, 1]^2 \rightarrow [0, 1]$, where $v(s_i, x)$ is i's valuation given the realizations (s_i, x) of i's signal and of the state variable.

Unless otherwise stated, we assume that the payoff to a bidder from consuming an amount q_i is

$$v(s_i, x)q_i.$$

At some points, we also consider a case where agents become satiated beyond some $\bar{q}_i$. In the example at the end of the paper, we illustrate some issues that arise when preferences take a richer form.[3]

In this setting, X represents an objective quality of the good or part that is common to all bidders; and the signal S_i has a dual role: it contains information regarding X and also represents a personal taste (see Milgrom (1981)). This framework includes as special cases settings of pure private values, where $v(s_i, x) = s_i$, and pure common values, where $v(s_i, x) = x$.

The following condition requires that at least one of the variables be important, and that there is some upper bound on the derivative with respect to the private signal.

Assumption A(II). $v(s_i, x)$ is differentiable and non-decreasing in both variables, and there exist $\rho > 0$ and $\gamma > 0$ such that $\partial v/\partial s_i + \partial v/\partial x > \rho$ and $\partial v/\partial s_i < \gamma$.

Condition A(II) is important in implying that signals have some importance, either directly in terms of private values and/or indirectly in providing some information about preferences through the common component. The upper bound also puts some limit on how sensitive preferences are to information.

2.4. A private value component

We sometimes refer to situations in which there is some private value component to the valuation. This does not have to be a case of pure private values, but is captured by the following definition.

A good has a *private value component* if there exists $\tau > 0$ such that $\partial v/\partial s_i(s_i, x) > \tau$ for all (s_i, x).

2.5. Mechanisms

Invoking the well-known revelation principle, we restrict attention to direct mechanisms.[4] A mechanism in the n-th economy is a pair of functions (q^n, t^n), where

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[3] The main results extend to cases where the valuation takes a form $v(s_i, x)h(q_i)$, for a non-decreasing and concave h, which incorporates the example. However, the simpler formulation provides for a more transparent exposition.

[4] Under our information and preference assumptions, there exists a (symmetric) equilibrium for a wide variety of auction formats (including all the standard ones) if the tie-breaking is allowed to be endogenous (see Jackson

1. $q^n : [0,1]^n \to \mathbb{R}^n$ is an allocation rule that assigns quantities to bidders as a function of the profile of announced signals s, such that $\sum_{i=1}^{n} q_i^n(s) \le Q^n$, and
2. $t^n : [0,1]^n \to \mathbb{R}_+^n$ is a payment function that specifies the payment each bidder makes as a function of the profile of announced signals s, where $t_i(s)$ denotes the payment of bidder i.

We make three remarks about the allocation and payment functions. First, the allocation function does not necessarily allocate all of Q^n. This allows for the incorporation of reserve prices into the auctions considered. Second, t_i^n can be positive even when q_i^n is not. Thus, the specification allows for bidders to pay even when they do not receive any allocation, and so it allows for features such as entry fees and "all-pay" requirements. Third, q_i^n and t_i^n can be thought of as expected allocations and payments so that randomization is permitted.

In some of the examples we consider mechanisms that treat bidders symmetrically. Most of the results, however, apply to asymmetric mechanisms. We explicitly note when symmetry is assumed.

Bidders are risk neutral, and so under a mechanism (q^n, t^n) the expected surplus (or payoff) obtained by bidder i who has a signal s_i and declares s_i' is

$$\mathrm{Sur}_i^n(s_i, s_i') = E[q_i^n(s_i', S_{-i})v(s_i, X) - t_i^n(s_i', S_{-i})|S_i = s_i].$$

Note that Sur_i^n is a function of the mechanism (q^n, t^n), but we suppress this notation as the mechanisms will usually be given.

Incentive compatibility is written as

$$\mathrm{Sur}_i^n(s_i, s_i) \ge \mathrm{Sur}_i^n(s_i, s_i')$$

for each i, s_i, and s_i'.[5]

2.6. Participation and self-enforcing mechanisms

A key ingredient in our analysis are participation constraints. We focus on mechanisms such that agents weakly desire to participate after observing the terms of trade, that is, after learning the quantity they will receive and the price they will pay. We name such mechanisms *self-enforcing*. Formally a *self-enforcing* mechanism is one where for any agent i and almost every profile of signals s,

$$q_i^n(s)E(v(s_i, X)|S_i = s_i, t_i = t_i^n(s), q_i = q_i^n(s)) \ge t_i^n(s).$$

To gain a better understanding of this condition, it is useful to compare it to an interim individual rationality constraint and an ex-post individual rationality constraint.

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et al. (2002)), and even with fixed tie-breaking for the case of private and possibly correlated values (see Jackson and Swinkels (2001)). Regardless of whether one deals with a pure or mixed strategy equilibrium or the nature of the tie-breaking, the corresponding direct mechanism is handled by our approach in this paper, and so the results here apply.

[5] As usual, all conditions are required to hold only almost surely and we omit such mention in what follows.

The interim individual rationality constraint implies that agents who have observed their signals weakly desire to participate. This constraint can be expressed as:

$$E(q_i^n(s_i, S_{-i})v(s_i, X)|S_i = s_i) \geq E(t_i^n(s_i, S_{-i})|S_i = s_i),$$

for all agents i and almost all signals s_i.

Interim individual rationality is weaker than self-enforcement as agents do not condition on the realization of their payment and allocation. A critical difference is that the mechanisms proposed by Crémer and McLean (1988)are interim individual rational but are not self-enforcing.

An ex-post individual rationality constraint is such that agents weakly desire to participate after observing *all* private information. It can be expressed as

$$q_i^n(s)E(v(s_i, X)|S = s) \geq t_i^n(s),$$

for all agents i and almost all profiles of signals s.

Ex post individual rationality is a stronger constraint than self-enforcement as agents condition on all private information and not just what they directly observe. Apart from the pure private values case, standard auction formats fail to be ex-post individually rational. To see this, consider a standard sealed-bid first price auction, with a pure common value to the object and where high signals lead to higher expectations regarding the common value. In the event where $n - 1$ agents get the lowest possible signal while one agent observes the highest possible signal then the winner regrets his participation after having learned all of the signals. A similar phenomena happens in a second price auction when $n - 2$ get the lowest possible signal and two agents observe the highest possible signal.

In sum, self-enforcement is stronger than an interim individual constraint and weaker then an ex-post participation constraint. It has the advantage of accommodating auction formats used in practice while relying on information agents directly observe.

The justification for self-enforcement depends on the ability of bidders to walk away from the mechanism, or to back out of making a payment, once they have learned the terms of the mechanism. Self-enforcing auctions are such that the bidders will be willing to make the prescripted payment and accept the object, once these terms of trade are realized.

2.7. *Concentrated versus dispersed allocations*

Let $\bar{q}_n \equiv Q_n/n$ denote the per-capita supply of objects for sale.

2.7.1. *Concentrated allocations*

A sequence of allocation functions $\{q^n\}$ is *concentrated* if for every $b > 0$ there exists an n' such that for $n > n'$ and every i

$$\text{Prob}\left(\left\{E\left[\frac{q_i^n(S)}{\bar{q}_n}|S_i\right] \geq b\right\}\right) < b.$$

2.7.2. *Dispersed allocations*

A sequence of allocation functions $\{q^n\}$ is *dispersed* if there exists $b > 0$ such that for infinitely many n

$$\mathrm{Prob}\left(\left\{E\left[\frac{q_i^n(S)}{\bar{q}_n}\,|S_i\right]\geq b\right\}\right)\geq b,$$

for a number of agents i that is at least bn.

In what follows, we refer to a sequence of mechanisms as being concentrated or dispersed if their corresponding allocation functions are.

The intuition behind these definitions is that a completely evenly dispersed allocation would give $q_i^n = \bar{q}_n$ to each bidder, so that $q_i^n/\bar{q}_n$ would be 1. If, instead, this expression is going to zero for almost all bidders, then the allocation is concentrated in the hands of just a small proportion of the bidders (i.e., those who saw certain signals), while if it is not vanishing for some non-trivial proportion of bidders (and so one can expect to get objects conditional on seeing a non-trivial range of signals) then there is reasonable dispersion.

Let us make a few remarks about the details of the definitions.

First, they are defined relative to the per-capita supply of the good. It is possible to have a dispersed allocation even if $\bar{q}_n \to 0$ and each bidder's allocation is actually going to 0. Similarly, it is possible to have a concentrated allocation even when $\bar{q}_n \to \infty$ and where every bidder is getting an arbitrarily large allocation in the limit, but the highest signal bidders are getting the lion's share.[6] So the important intuition regarding competition that emerges here is that it is the *relative* disparities in allocations that determine whether or not surplus is competed away in large auctions.

Second, the definitions allow for asymmetric mechanisms, so that different bidders might have different expectations under the mechanisms in question. However, it is important to note that concentration requires a uniformity in the convergence across bidders i. Without this, it would be possible, for example, to have non-vanishing fractions of bidders expecting to get significant fractions of the objects at any date. For example, suppose that objects are simply randomly given to agents with labels between $n/2$ and n in auction n. Here, any given bidder eventually expects to get no objects at all, and yet the allocation is clearly not what one would want to call "concentrated." Thus, the uniformity in convergence rates across bidders under the definition of concentration is critical.

Third, there is a gap between the definitions of concentrated and dispersed allocations, and some sequences of mechanisms do not fall into either category. This gap is necessary given that we wish to account for asymmetric mechanisms. For instance, consider the following situation. There are n bidders and the object is always simply given to bidder 1 at a price of 0.[7] This clearly fails to be a concentrated sequence under the definition. This is important, because the results that are claimed for sequences of concentrated mechanisms (e.g., bidders' surplus going to 0) would not be true for this particular sequence of asymmetric mechanisms. Note also, that it would not make sense to call this a "dispersed allocation," as the objects are always going to one bidder. The reason that our results do not apply is that the particular

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[6] These features make the conditions different, for instance, from checking whether Q^n and $n - Q^n$ are getting large as in the double-largeness condition of Pesendorfer and Swinkels (1997). In fact, $n - Q^n$ does not play any role in our analysis, and Q^n only plays a role in the denominator in determining relative allocations. This means that there are some differences between conditions that ensure information aggregation, and those which correspond to surplus extraction.

[7] This is not such a silly mechanism, as note that it corresponds to an asymmetric equilibrium in a second price auction where bidder 1 always bids 1 regardless of her signal and all other bidders always bid 0.

asymmetry in the mechanism has eliminated all competition for one bidder. Thus, allowing for asymmetric mechanisms requires some gap between the definitions. Note, however, that if one restricts attention to symmetric mechanisms, then the definitions are essentially complementary.

Finally, whether the allocation is concentrated or dispersed depends on the setting and the equilibrium that will result in a given sequence of auctions. Looking at things through the lens of the allocation allows us to extract the general insight regarding how competitive forces work in large economies and how this depends on the distribution of goods. Nevertheless, it is important to be able to tell which type of allocation applies in different situations. The classification of whether or not the allocation turns out to be concentrated or dispersed is often straightforward. That is, many situations can be categorized into general classes where it is clear which type of allocation will result under most standard auction formats. A simple classification is as follows.

- A concentrated allocation will necessarily result if the quantity of good to be allocated (Q^n) is a vanishing fraction of n, the good is (asymptotically and approximately) efficiently allocated, and there is some private component to the valuation.
- A dispersed allocation will necessarily result if bidders have a finite bound on the amount of the good that they desire and the amount of the good grows in proportion to n.

The above only provides a rough classification, but still covers many of the cases of interest. Auctions of limited numbers of objects (e.g., an art auction) will generally fall into the first case and have a concentrated allocation, while auctions of many objects (e.g., treasury auctions) will often fall into the second case and have dispersed allocations. Given the variety of mechanisms and settings admitted in the model, a fuller characterization of when concentrated versus dispersed allocations result would be quite complicated, without adding much insight. We provide a fuller treatment of two prominent auction formats (uniform and discriminatory) in Jackson and Kremer (2003). We now turn to analyzing auctions under the two types of allocations.

3. Concentrated allocations

We first examine the implications of implementing concentrated allocations. In concentrated allocations, for any bidder who gets a significant proportion of the good there exists another bidder who receives a nearby signal (and hence has nearby beliefs and valuation), but only gets a relatively small amount of the good. The competition from such nearby bidders eliminates the surplus enjoyed by all bidders.

Theorem 1. *If* A(I) *and* A(II) *hold, then for any sequence of concentrated, interim individually rational,*[8] *and incentive compatible mechanisms,* $\left(\sum_i \mathrm{Sur}_i^n(S_i, S_i)\right)/Q^n$ *converges to 0 in probability.*[9]

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[8] Interim individual rationality implies that the same holds for self-enforcing mechanisms.

[9] We actually prove that the *expected* per unit surplus converges to 0, which implies convergence in probability since this is a nonnegative random variable.

The intuition behind the theorem is as follows. Under a concentrated allocation, the circumstances in which a bidder can expect to win non-trivial amounts of objects (in per-capita terms) is shrinking. That is, the set of signals under which a bidder expects to win objects is a shrinking set. Nearby signals must lead to expectations of no surplus. Then given incentive compatibility and the fact that nearby signals lead to nearly the same expectations, since nearby signals expect a low surplus, the winning signals must also expect a low surplus. In terms of more traditional language of competition: the objects are being concentrated in the hands of just a few winning bidders. As the economy grows, there will also be many other bidders who have very similar information and preferences to those who end up winning. The competition between these bidders eliminates the surplus.

3.1. Optimal mechanisms

Theorem 1 implies that the total revenue in a sequence of auctions with concentrated allocation functions is the approximate full (expected) valuation of the objects to the winning bidders. As we have not specified the allocation functions beyond being concentrated, this does not necessarily imply full revenue extraction. For instance, it could be that the mechanisms never give any objects away and do not result in any revenue. A sufficient condition for full revenue extraction is that the efficient allocation be concentrated, as then Theorem 1 implies that any sequence of auctions that results in efficient allocations provides full revenue extraction in the limit. One implication of this is that with large numbers of bidders, auction formats that lead to efficient and concentrated allocations also lead to approximately full revenue and one does not need to resort to the complicated and parametric types of mechanisms identified by Crémer and McLean (1988) and McAfee and Reny (1992). Moreover, this holds in a variety of settings, including correlated private values, common values, as well as under complete independence (where Crémer and McLean mechanisms fail to work).

To be careful, we have to argue that there exist mechanisms that achieve efficiency (at least approximately when efficient allocations are concentrated) in order for the above statements to be non-vacuous.[10] Indeed, there exist mechanisms, even symmetric ones, that will achieve an approximately efficient allocation in a variety of situations, and so full revenue extraction is feasible. This mechanism even satisfies ex-post individual rationality constraints and works without the correlation structure inherent in the Crémer–McLean approach. Without giving a formal argument, let us heuristically describe such mechanisms in the case where the efficient allocation involves awarding all objects to one bidder.[11,12] Pick some subset of agents and ask them their signals. If symmetry is desired, randomly pick the agents. Keep this set of surveyed agents of size $\sqrt{n}$, so that it grows with n, but is negligible in the limit. These agents will not get any of the allocation, so it is incentive compatible for them to reveal their information. Based on their announcements,

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[10] We thank a referee for pointing this out.

[11] Without any satiation it is efficient to treat the supply as indivisible. This can be modified as long as the efficient allocation is concentrated.

[12] Variations on this sort of “folk” mechanism appear in a number of places. For an auctions version, where interdependencies in valuations are present, see Jackson (2003). For versions satisfying strategy-proofness in private values settings, see Cordoba and Hammond (1998) and Kovalenkov (2002).

estimate X, and then $v(1, X)$. Randomly order the remaining agents, and make them take it or leave it offers at the price of $v(1, X) - \varepsilon$, until some agent agrees to buy the objects. This will happen with very high probability for large enough n, and the object(s) will end up in the hands of an agent who values them at nearly the maximal possible level.

3.2. Revenue equivalence

Theorem 1 provides a revenue equivalence result, in that any two sequences with similar allocation functions must result in similar revenues. This is stated in the following corollary.

Corollary 1. *Let* A(I), A(II) *hold and consider two sequences of incentive compatible and interim individually rational mechanisms,* $\{(q^n, t^n)\}$ *and* $\{(\hat{q}^n, \hat{t}^n)\}$*, with concentrated allocation functions. If the allocation functions are approximately the same, i.e.,*

$$\frac{E\left[\sum_i q_i^n(S)v(S_i, X) - \sum_i \hat{q}_i^n(S)v(S_i, X)\right]}{Q^n} \to 0,$$

then they lead to approximately the same expected revenues:

$$\frac{E\left[\sum_i t_i^n(S) - \sum_i \hat{t}_i^n(S)\right]}{Q^n} \to 0.$$

Corollary 1 provides a fairly general asymptotic revenue equivalence theorem, as it applies with correlated values and/or common values, and the sale of more than one object.[13]

As we shall see, it is critical to the above result that the mechanisms be concentrated. Otherwise, mechanisms with identical allocation functions can lead to very different revenues, even in the limit.

3.3. Indivisible goods

A case of concentrated allocations that is of particular interest is where an indivisible good is to be auctioned to the highest bidder. This covers, for instance, first price, second price, and English auctions. We focus on this case to get some insight into rates of convergence to the competitive outcome. We first show that the surplus to the winning bidder decreases at a faster rate than $1/n^a$ for any $a < 1$. To develop a tight bound on surplus, we also consider the following condition on the information structure.

Assumption A(III). There exists $\beta > 0$ such that $\beta > f(s_i|x) > 1/\beta$ for every s_i and x.

A(III) bounds the likelihood of any signal conditional on a given X both above and below, thus limiting the informativeness of any given signal and implying some diversity in the signals observed.

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[13] Corollary 1 generalizes the main result of Bali and Jackson (2002), in that it applies to the auctioning of more than one good, and also allows for entry fees. However, it requires more structure on information (the mineral rights setting) and on mechanisms than the results of Bali and Jackson (2002).

Theorem 2. *Let* A(I)–A(III) *hold and consider a sequence of incentive compatible and interim individually rational mechanisms,* $\{(q^n, t^n)\}$*, which award the entire* Q^n *to a highest signal observer. The bidders' surplus (per unit) converges to zero at a rate faster than* $1/n^a$ *for any* $a < 1$*. That is for any* $a < 1$, $n^a \sum_i \mathrm{Sur}_i^n(S_i, S_i)/Q^n$ *converges to 0 in probability.*

The bound in Theorem 2 comes from bounding the conditional probability of winning for an observer of a given signal. That probability goes to 0 at an exponential rate, even for signals going to 1 at a rate n^a ($a < 1$). This bounds the surplus that can be expected for high signals. The continuity of information then implies that this surplus is approximately the same as is enjoyed by the highest signal. The complete proof appears in the Appendix A.

We now explore the tightness of this bound. We show that the surplus going to bidders is at least the order of $1/n$ in any case where the good has some private value component, and so the bounds established in Theorem 2 are tight for a standard class of auctions.

In order to obtain this bound, we need to have some handle on the structure of bidding, and to this end we add an assumption of affiliated signals in the usual form of the Monotone Likelihood Ratio Property, so that higher signals relate to higher expected values of the common parameter X.

Assumption A(IV)

$$f(s_i|x)f(s_i'|x') \geq f(s_i'|x)f(s_i|x') \text{ for all } s_i > s_i' \text{ and } x > x'.$$

Theorem 3. *Let* A(I)–A(IV) *hold and consider a good that has a private value component and a sequence of self-enforcing mechanisms* $\{(q^n, t^n)\}$*, which award the entire* Q^n *to a highest signal observer. There exists* $\phi > 0$ *such that the total surplus per unit to the bidders is at least* ϕ/n *for all* n*. That is, there exists* $\phi > 0$ *such that for any* n

$$\frac{E\left[\sum_i \mathrm{Sur}_i^n(S_i, S_i)\right]}{Q^n} > \frac{\phi}{n}.$$

Theorem 3 shows that the bound established in Theorem 2 is tight. It is proven by showing that the winner expects a distance between her signal and the next highest signal that is on the order of $1/n$. This implies, given the private value component, that the winner expects to have a valuation that is higher than the second highest by an amount that is of the order of $1/n$. Then, regardless of the particular payment format, incentive compatibility implies that winner must get a surplus of the order of $1/n$.

If we allow for arbitrary mechanisms, then with correlation among the signals there is a possibility of extracting full surplus from the bidders, as shown by Crémer and McLean (1988)[14]. The full extraction mechanisms are ruled out under Theorem 3 as payments never exceed the maximum possible value and are only made conditional on receiving the object. Neither of these conditions are met by the Crémer–McLean style mechanisms, as such mechanisms require occasionally large payments and payments even by bidders who do not obtain the object. These features of Crémer–McLean style mechanisms are not exhibited by

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[14] McAfee and Reny (1992) show that this is also true for the continuous signal case, to an arbitrary approximation.

many standard auction formats (e.g., first price, second price, English auctions, etc.) which satisfy the condition of Theorem 3.

3.4. An application to endogenous entry

We now show that the results regarding the convergence rate of bidders' surplus are not simply a technical curiosity, but can be used to provide insight into auctions where the entry decision is endogenous and costly.[15]

Suppose that a quantity of the good Q is sold as an indivisible good, and bidders must pay an (ex-ante) entry fee of c.[16] Let us examine the number of entrants and the markdown in prices as a function of Q and entry cost, c. The novelty is that we establish these relations without relying on a specific mechanism. We only assume that mechanism does not charge a bidder unless they get the good (excluding the entry cost), payment never exceeds the upper bound on the good's value, and the good has some private value component (we also assume the information assumptions from the last section).

Theorem 3 tells us that there exists some $\phi > 0$ such that the total surplus that goes to bidders exceeds $\phi Q/n$ for any n. Hence, in order for it to be an equilibrium for n and not $n+1$ bidders to enter (at an ex-ante stage before signals are observed) we know that

$$c \geq \frac{1}{n+1}\left(\frac{\phi Q}{n+1}\right)$$

or that

$$n \geq \sqrt{\frac{\phi Q}{c}} - 1.$$

This gives us a lower bound on n. Next, let us develop an upper bound. If Q/c (and hence n) is large, Theorem 2 bounds the total surplus to be below Q/n^a for any $a < 1$. Thus, for n bidders to be willing to enter we must have

$$\left(\frac{1}{n}\right)\frac{Q}{n^a} \geq c,$$

or

$$\left(\frac{Q}{c}\right)^{1/(1+a)} \geq n.$$

Putting these lower and (approximate) upper bounds together leads to

$$\left(\frac{Q}{c}\right)^{1/(1+a)} \geq n \geq \sqrt{\frac{\phi Q}{c}} - 1,$$

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[15] For other examples of usefulness of such convergence rates in auctions and bargaining see Rustichini et al. (1994) and Neeman (1999).

[16] We are considering a two stage process where bidders first decide whether to enter or not and then observe their signals and participate in the auction if they have paid the entry fee.

for any $a < 1$. So, we have obtained an approximation on the number of bidders who will enter an auction:[17]

$$n \propto \sqrt{\frac{Q}{c}}$$

Since the expected surplus excluding entry costs that goes to the bidders is on the order of Q/n (Theorems 2 and 3), substituting from the approximation for n we find that the expected surplus going to bidders in the auction is approximately proportional to $\sqrt{Qc}$. This in turn implies that the average markdown in price per unit (compared to the winner's valuation) is approximately proportional to $\sqrt{c/Q}$.

4. Dispersed allocations

We now turn our attention to sequences of auctions with dispersed allocations.

4.1. The impossibility of full surplus extraction

There is a major difference in behavior between mechanisms with dispersed versus concentrated allocations. The following theorem shows that the (approximate) full-extraction of revenue that occurs with concentrated allocations will not hold with dispersed allocations, regardless of the mechanism employed.

Theorem 4. *Suppose that* A(I)–A(IV) *hold, and consider a good that has a private value component and a sequence of incentive compatible and self-enforcing mechanisms with dispersed allocations. There exists* $\phi > 0$ *such that the expected total surplus per unit obtained by the bidders in the auction is at least* ϕQ^n *for infinitely many* n. *That is, there exists* $\phi > 0$ *such that for infinitely many* n

$$E\left[\sum_i \mathrm{Sur}_i^n(S_i, S_i)\right] \geq \phi Q^n.$$

Theorem 4 tells us that if the allocation is dispersed and there is any private component to the valuation structure, then bidders necessarily capture non- negligible rents, even in arbitrarily large economies.

We state the intuition for the case where allocations are to high bidders, but such monotonicity is not essential (see the proof for details). Under dispersed allocations, a bidder with a high signal could pretend to have a slightly lower signal and still expect with some non-trivial probability to obtain some of the object. As long as (i) the high signal bidder does not expect to pay more than what would be fair for a bidder for the lower signal (implicit in self-enforcement), and (ii) according to the high type's belief a lower type should expect to get a non-trivial fraction of the good (the role of dispersion), it follows that the high signal

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[17] The approximation, of course, is only valid for large n, and so is more accurate if the total value of goods to be auctioned relative to the entry cost (Q/c) is large.

bidder could obtain a positive expected surplus by pretending to have observed the lower signal. By incentive compatibility, the high-signal observing bidder must get at least this surplus under truthful announcement of his signal.

An implication of Theorem 4 is that the possibility of designing mechanisms that extract full revenue is precluded for the case where the efficient allocation is dispersed, at least under self-enforcement.

It is also interesting to note that approximate revenue equivalence no longer holds when the efficient allocation is dispersed. This is illustrated by Jackson and Kremer (2003) who provide a detailed comparison of Discriminatory and Uniform price auctions for the case of private values.

4.2. *Revenue versus efficiency*

While Theorem 4 shows that when the efficient allocation is dispersed full revenue extraction is not possible, it is still conceivable that in a large society, subject to allocating the goods, the seller would want to allocate them efficiently. We now point out, however, that even subject to allocating all of the goods, the seller may want to inefficiently allocate them, and hence there can be a very fundamental tension between efficiency and revenue maximization. For this, we look to a richer preference structure than that examined up to this point.

In auction design, there are several tensions between efficiency and revenue that have been noted in the literature. First, the commitment to a reservation fee can raise expected revenues while decreasing efficiency (e.g., see Myerson (1981)), as sometimes an object is not sold when it would be efficient to do so. Second, with asymmetric distributions of information, awarding the object to the bidder with the highest virtual utility (which maximizes revenue) may conflict with awarding the object to the bidder with the highest utility (which is efficient), as shown by Myerson (1981). Third, with heterogeneous objects, a seller may have an incentive to bundle objects together (Palfrey, 1983; Jehiel and Moldovanu, 2001). The example below points out that such an incentive to inefficiently allocate objects (in particular to bundle them) arises even in a case with homogeneous objects, independent symmetric type distributions, and subject to allocating all of the goods!

Example. Bundling

S_i is distributed uniformly on $[0, 1]$. There are $n/2$ indivisible objects to be allocated. Instead of having a constant marginal valuation up to some satiation point and then no value beyond that, let us consider a slightly richer structure. Bidders have a value of s_i for a first object, a value of $s_i/2$ for a second object, and no value for any additional objects.[18]

First, consider any (approximately) efficient allocation, which for large numbers corresponds (approximately) to giving one object to each of the $n/2$ highest signal observers. Given the independent signals, revenue equivalence among individually rational

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[18] As we pointed out earlier, while we specified a linear framework our results generalize to such decreasing return to scale framework.

mechanisms holds in this world (see Ausubel and Cramton (1995)), and so the expected revenue of any mechanism that results in this allocation converges to $1/2$ per object.

Next, consider the following inefficient allocation.[19]Objects are bundled and only sold in pairs. The pairs of objects are awarded to the $n/4$ highest signal holders via a Vickrey auction. In this case, the price setting bidder for large n will have a signal of approximately $3/4$. That bidder's valuation for a pair of objects will be $3/4 + (1/2)3/4 = 9/8$, and the revenue per object converges to $9/16$.

Although bundling leads to an inefficient allocation, it lead to an increase in revenue of over 6% compared to mechanisms leading to the efficient allocation.

In the case of dispersed allocations we know that the mechanism matters. Moreover, as the example shows, sellers might prefer auctions which bundle objects in inefficient ways. Obtaining a better understanding of optimal mechanisms in such situations, as well as the tension between efficiency and revenue maximization, is a challenging but important open problem.

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Appendix A

Lemma 1. *Under* A(I)–A(IV), *if a mechanism is self-enforcing then*

$$E[q_i^n(s_i, S_{-i})v(s_i, X) - t_i^n(s_i, S_{-i})|S_i = s'_i] \geq 0,$$

for any i and almost every s_i and s_i' such that $s'_i \geq s_i$.

Proof of Lemma 1. By the law of iterated expectations

$$\begin{aligned} &E[q_i^n(s_i, S_{-i})v(s_i, X) - t_i^n(s_i, S_{-i})|S_i = s'_i] \\ &\quad = E[q_i^n(s_i, S_{-i})E[v(s_i, X)|S_i = s'_i, q_i^n(s_i, S_{-i}), t_i^n(s_i, S_{-i})] - t_i^n(s_i, S_{-i})|S_i = s'] \end{aligned}$$

for almost every s_i and s_i'.[20]

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[19] As shown by Ausubel and Cramton (1999), a perfect resale market for goods leads to a revenue maximizing mechanism being an efficient one. So, it is important that perfect (costless) resale is not possible in this example.

[20] Note that under A(I) and A(III), the supports of S_{-i} conditional on s_i and conditional on s_i' are the same, and so conditioning on $S_i = s_i'$, $q_i^n(s_i, S_{-i})$, $t_i^n(s_i, S_{-i})$ is not a problem.

A(IV) implies that for almost every s_i and s'_i

$$E[v(s_i, X)|S_i = s'_i, q_i^n(s_i, S_{-i}), t_i^n(s_i, S_{-i})] \geq E[v(s_i, X)|S_i = s_i, q_i^n(s_i, S_{-i}), t_i^n(s_i, S_{-i})],$$

almost surely. Thus,

$$E[q_i^n(s_i, S_{-i})v(s_i, X) - t_i^n(s_i, S_{-i})|S_i = s'_i] \geq E[q_i^n(s_i, S_{-i})E[v(s_i, X)| S = s_i, q_i = q_i^n(s_i, S_{-i}), t_i = t_i^n(s_i, S_{-i})] - t_i^n(s_i, S_{-i})|S_i = s'_i]. \quad (A.1)$$

By self-enforcement, it follows that for almost every s_i,

$$q_i^n(s_i, S_{-i})E[v(s_i, X)|S = s_i, q_i = q_i^n(s_i, S_{-i}), t_i = t_i^n(s_i, S_{-i})] - t_i^n(s_i, S_{-i}) \geq 0$$

for almost every realization of S_{-i}. Thus, it follows from (A.1) that

$$E[q_i^n(s_i, S_{-i})v(s_i, X) - t_i^n(s_i, S_{-i})|S_i = s'_i] \geq 0,$$

as claimed. □

The following “continuity” lemma is useful. It states that the surplus obtained by a given type is nearly obtainable by a nearby type who pretends to be of the given type.

Lemma 2. *If* A(I) *and* A(II) *are satisfied, then in any sequence of interim individually rational mechanisms and for any n*

$$|\mathrm{Sur}_i^n(s_i, s'_i) - \mathrm{Sur}_i^n(s'_i, s'_i)| \leq (2\alpha + \gamma)|s_i - s'_i|E^n[q_i^n(S)|S_i = s'_i],$$

for all i and for almost every s_i, s'_i, for some $\alpha > 0$ and $\gamma > 0$.

Proof of Lemma 2. Write

$$I^n + II^n \geq |\mathrm{Sur}_i^n(s_i, s'_i) - \mathrm{Sur}_i^n(s'_i, s'_i)|$$

where I^n is the difference in the utility from the good received:

$$I^n = \left| \int q_i^n(s'_i, s_{-i})v(s_i, x)\, \mathrm{d}F^n(s_{-i}, x|s_i) - \int q_i^n(s'_i, s_{-i})v(s'_i, x)\mathrm{d}F^n(s_{-i}, x|s'_i) \right|$$

and II^n is the difference in expected payment

$$II^n = \left| \int t_i^n(s'_i, s_{-i})\, \mathrm{d}F^n(s_{-i}|s_i) - \int t_i^n(s'_i, s_{-i})\mathrm{d}F^n(s_{-i}|s'_i) \right|.$$

Step 1. $I^n < (\alpha + \gamma)|s_i - s'_i|E[q_i^n(S)|S_i = s'_i]$

A(II) ($\mathrm{d}v/\mathrm{d}s_i < \gamma$) implies that for any n

$$\left| \int q_i^n(s'_i, s_{-i})[v(s_i, x) - v(s'_i, x)]\, \mathrm{d}F^n(s_{-i}, x|s'_i) \right| \leq \gamma|s_i - s'_i|E[q_i^n(S)|S_i = s'_i]. \quad (A.2)$$

A(I) implies that:

$$|\mathrm{d}F^n(s_{-i}|s_i) - \mathrm{d}F^n(s_{-i}|s_i')| = \left|\int_x f^n(s_{-i}|x)(f(x|s_i) - f(x|s_i'))\,\mathrm{d}x\right|$$
$$\leq \alpha|s_i - s_i'|\mathrm{d}F^n(s_{-i}|s_i').$$

Then since $1 \geq |v(s,x)|$, we deduce that:

$$\left|\int q_i^n(s_i', s_{-i})v(s_i,x)\,\mathrm{d}F(s_{-i},x|s_i) - \int q_i^n(s_i', s_{-i})v(s_i,x)\,\mathrm{d}F^n(s_{-i},x|s_i')\right| \tag{A.3}$$
$$\leq \alpha|s_i - s_i'|E^n[q_i^n(S)|S_i = s_i'].$$

Hence, the claim in Step 1 follows from (A.2) and (A.3).

Step 2. $II^n \leq \alpha|s_i - s_i'|E^n[q_i^n(S)|S_i = s_i']$

By A(I) it follows that

$$\left|\int t_i^n(s_i', s_{-i})\,\mathrm{d}F^n(s_{-i}|s_i) - \int t_i^n(s_i', s_{-i})\,\mathrm{d}F^n(s_{-i}|s_i')\right| \leq \alpha|s_i - s_i'|E^n[t_i^n(S)|S_i = s_i'],$$

for some $\alpha > 0$. Thus, $II^n < \alpha|s_i - s_i'|E^n[t_i^n(S)|S_i = s_i']$. From the interim individual rationality constraint and the fact that $1 \geq v(s_i, x)$, it follows that:

$$E^n[q_i^n(S)|S_i = s_i'] \geq E^n[t_i^n(S)|S_i = s_i'].$$

which concludes the argument. □

Proof of Theorem 1. We actually prove that

$$\frac{E\left(\sum_i \mathrm{Sur}_i^n(S_i, S_i)\right)}{Q^n} = \sum_i \int_{[0,1]} \frac{\mathrm{Sur}_i^n(s,s)}{Q_n}\mathrm{d}F(s),$$

converges to 0, which implies convergence in probability since $\mathrm{Sur}_i^n(S_i, S_i)$ is a nonnegative random variable.

Fix any small ε and by Lemma 2 find a δ such that $|s - s^*| < \delta$ iimplies that for all i and n

$$|\mathrm{Sur}_i^n(s^*, s) - \mathrm{Sur}_i^n(s,s)| \leq \varepsilon E^n[q_i^n(S)|S_i = s].$$

For the given ε, let $A_i^n(\varepsilon)$ denote the set of types who expect to receive supply less than $\varepsilon\bar{q}_n$, that is:

$$A_i^n(\varepsilon) = \left\{s_i \quad \text{s.t.} \quad E\left[\frac{q_i^n(S)}{\bar{q}_n}|S_i = s_i\right] < \varepsilon\right\}.$$

Pick $n > n'$ (where n' is defined by concentration) so that for any s that is in the support of f, there exists $s_i^n(s) \in A_i^n(\varepsilon)$ such that $|s - s_i^n(s)| < \delta$. Incentive compatibility and Lemma 2 then imply that:

$$\mathrm{Sur}_i^n(s,s) \leq \mathrm{Sur}_i^n(s_i^n(s), s_i^n(s)) + \varepsilon E^n[q_i^n(S)|S_i = s] \tag{A.4}$$

We can bound the surplus of types $s_i^n(s) \in A_i^n(\varepsilon)$:

$$\sum_i \frac{\mathrm{Sur}_i^n(s_i^n(s), s_i^n(s))}{Q_n} \leq \sum_i E\left[\frac{q_i^n(S)}{Q_n} | S_i = s_i^n(s)\right] < \varepsilon \tag{A.5}$$

Thus, from (A.4) and (A.5) it follows that for large enough n

$$\sum_i \int_{[0,1]} \frac{\mathrm{Sur}_i^n(s,s)}{Q_n} \mathrm{d}F(s) \leq \varepsilon + \sum_i \int_{[0,1]} \varepsilon \frac{E^n[q_i^n(S)|S_i = s]}{Q_n} \mathrm{d}F(s). \tag{A.6}$$

Since

$$Q_n \geq \sum_i \int_{[0,1]} E^n[q_i^n(S)|S_i = s] \mathrm{d}F(s),$$

it follows that

$$1 \geq \sum_i \int_{[0,1]} \frac{E^n[q_i^n(S)|S_i = s]}{Q_n} \mathrm{d}F(s). \tag{A.7}$$

(A.6) and (A.7) then imply that for large enough n

$$\sum_i \int_{[0,1]} \frac{\mathrm{Sur}_i^n(s,s)}{Q_n} \mathrm{d}F(s) \leq 2\varepsilon.$$ □

The following lemma is useful in the proof of Theorem 2.

Lemma 3. *Let* $s^n = 1 - n^{a-1}$. *If* A(III) *is satisfied, then there exists some* $b > 0$ *and some* N *such that* $F_{Y_{-i}^n(1)|S_i}(s^n | s_n^i) < \mathrm{e}^{-bn^a}$ *for all* $n > N$.

Proof of Lemma 3. The claim is clear if $a \geq 1$, so consider $a < 1$. Write

$$F_{Y_{-i}^n(1)|S_i}(s^n | s_n^i) = \int_x F_{Y_{-i}^n(1)|X}(s^n | x) f(x | s_i^n) \mathrm{d}x.$$

By A(III) it follows that for any x

$$F_{Y_{-i}^n(1)|X}(s^n | x) < \left(1 - \frac{n^{a-1}}{\beta}\right)^{n-1}.$$

Thus,

$$F_{Y_{-i}^n(1)|S_i}(s^n | s_n^i) < \left(1 - \frac{n^{a-1}}{\beta}\right)^{n-1}.$$

Since $(1 - n^{a-1}/\beta)^{n-1} \to \mathrm{e}^{-n^a/\beta}$, the claim follows. □

Proof of Theorem 2. First, we show that for any $a > 0$, there exists N' such that for all s_i and all $n > N'$

$$\mathrm{Sur}^n(s_i, s_i) < 2\alpha n^{a-1} Q^n, \tag{A.8}$$

where α is identified in A(I).

Let $s^n = 1 - n^{a-1}$, and identify b and N from Lemma 3, such that $F_{Y^n_{-i}(1)|S_i}(s^n|s^i_n) < \mathrm{e}^{-bn^a}$ for all $n > N$.

If $s_i < s^n$ for some $n > N$, then it follows that $\mathrm{Sur}^n(s_i, s_i) < Q^n \mathrm{e}^{-bn}$. Taking, N'' to be large enough so that $\mathrm{e}^{-bn^a} < \alpha n^{a-1}$, we know that (A.8) holds for any $s_i < s^n$ when $n > N' = \max\{N, N''\}$.

Next, consider any $s_i \geq s^n$ for some $n > N'$. By Lemma 2

$$|\mathrm{Sur}^n(s^n, s_i) - \mathrm{Sur}^n(s_i, s_i)| < \alpha n^{a-1} Q^n.$$

Since $\mathrm{e}^{-bn} Q^n \geq \mathrm{Sur}^n(s^n, s^n) \geq \mathrm{Sur}^n(s^n, s_i)$, this implies that

$$\mathrm{Sur}^n(s_i, s_i) < \mathrm{e}^{-bn} Q^n + \alpha n^{a-1} Q^n.$$

Since $n > N'$ we know that $\mathrm{e}^{-bn^a} < \alpha n^{a-1}$ (recall the definitions of N'' and N'), and so (A.8) holds for any $s_i \geq s^n$. Thus, we have established (A.8).

So, let us now argue that for any $a > 0$, $n^{1-a} \sum_i \mathrm{Sur}^n_i(S_i, S_i)/Q^n$ converges to 0 in probability.

Since Q^n goes to just one bidder,[21]

$$\int \mathrm{Sur}^n(s_i, s_i) \mathrm{d}F_{Y^n(1)}(s_i) \geq E\left[\sum_i \mathrm{Sur}^n(S_i, S_i)\right].$$

So, from (A.8) it follows that for any $a > 0$ there exists N such that for any $n > N$

$$E\left[\sum_i \mathrm{Sur}^n(S_i, S_i)\right] < 2\alpha n^{a-1} Q^n. \tag{A.9}$$

Let us verify that this implies the theorem. First, we show that $(n^{1-a} E\left[\sum_i \mathrm{Sur}^n_i(S_i, S_i)\right])/Q^n$ converges to 0. Suppose the contrary. Then there exists $a' > 0$ and $\delta > 0$ such that

$$E\left[\sum_i \mathrm{Sur}^n(S_i, S_i)\right] > \delta n^{a'-1} Q^n$$

for infinitely many n. Taking $a < a'$, this violates (A.9)for some large enough n. Thus, our supposition was incorrect and so $(n^{1-a} E[\sum_i \mathrm{Sur}^n_i(S_i, S_i)])/Q^n$ converges to 0. Since $\mathrm{Sur}^n_i(S_i, S_i) \geq 0$, it follows that for any $a > 0$, $(n^{1-a} \sum_i \mathrm{Sur}^n_i(S_i, S_i))/Q^n$ converges to 0 in probability. □

Proof of Theorem 3. We bound $n\mathrm{Sur}^n(s_i, s'_i)/Q^n$ from below (across n) for a bidder observing some $s_i > 1 - (a/n)$ and reporting $s'_i = 1 - (2a/n)$, for some $a > 0$. By incentive compatibility, this gives a lower bound on $n\mathrm{Sur}^n(s_i, s_i)/Q^n$. We then show that there is a probability bounded from below that a winning bidder observes such a signal, which then implies that $nE[\sum_i \mathrm{Sur}^n(S_i, S'_i)]/Q^n$ is bounded below.

---

[21] This inequality needs not hold with equality, since it may be that payments are made by losing bidders.

So, let us show that $n\mathrm{Sur}^n(s_i, s_i')/Q^n$ is bounded below for a bidder observing some $s_i > 1 - (a/n)$ and reporting $s_i' = 1 - (2a/n)$, for some $a > 0$.

$$\mathrm{Sur}(s_i, s_i') = \int q_i^n(s_i', s_{-i})v(s_i, x) - t_i^n(s_i', s_{-i})\mathrm{d}F^n(x, s_{-i}|s_i) = I^n + II^n$$

where,

$$I^n = \int q_i^n(s_i', s_{-i})[v(s_i, x) - v\left(s_i', x\right)]\mathrm{d}F^n(s_{-i}, x|s_i)$$

and

$$II^n = \int [q_i^n(s_i', s_{-i})v(s_i', x) - t_i(s_i', s_{-i})]\mathrm{d}F^n(x|s_i)$$

We first note that Lemma 1 implies that $II^n \geq 0$. We now examine I^n. Since the good has a private value component, $v(s_i, x) - v(s_i', x) > \tau a/(2n)$. This implies that:

$$I^n > Q_n \frac{\tau a}{2n} F_{Y^n_{-i}(1)|S_i}(s_i'|s_i). \tag{A.10}$$

Given the state $X = x$ signals are independent, hence:

$$F_{Y^n_{-i}(1)|S_i,X}(s_i'|s_i, x) = F_{Y^n_{-i}(1)|X}(s_i'|x) = F_{S_i|X}(s_i'|x)^{n-1}$$

Using Assumption A(III), we conclude that $F_{S_i|X}(s_i'|x) > 1 - (2a\beta)/n$ for all x which implies that that there exists some $\alpha^* > 0$ so that[22]

$$F_{Y^n_{-i}(1)|X}(s_i'|x) > \alpha^*$$

Thus, since $F_{Y^n_{-i}(1)|S_i}(s_i'|s_i) = \int_x F_{Y^n_{-i}(1)|X}(s_i'|x)\mathrm{d}f(x|s_i)$, it follows from (A.10) that $I^n/(Q^n/n)$ is bounded below.

To complete the proof, we need to show that the probability that the winning signal is larger than $1 - (a/n)$ is bounded below. Again, A(III) implies that $F_{S_i|X}(1 - (a/n)|x) < 1 - (a/\beta n)$ for all x and $s_i > 1 - (a/n)$. Thus, $F_{Y^n(1)|X}(1-)a/n)|x) < (1 - (a/\beta n))^n$, which converges to $\mathrm{e}^{-a/\beta}$. So, there is a probability bounded below that the winning signal exceeds $1 - (a/n)$. □

Proof of Theorem 4. We need only prove the theorem for symmetric mechanisms. The extension to asymmetric mechanisms is then seen rather simply. Suppose to the contrary that some sequence of dispersed asymmetric mechanisms leads to an expected surplus heading to zero. Construct a sequence of symmetric mechanisms by randomly labeling the agents in the n-th mechanism. This must lead to the same expected total surplus, and is still incentive compatible, self-enforcing, and dispersed. But this would contradict the fact that the result holds for symmetric mechanisms.

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[22] The expression $(1 - (2a\beta/n))^{n-1}$ converges to $\mathrm{e}^{-2a\beta}$.

By dispersion, there exists $\varepsilon > 0$, $a > 0$, and for each n (taking a subsequence if necessary) a signal $s_i^n < 1 - 3\varepsilon$ such that

$$E(q_i^n(s_i^n, S_{-i})|S_i = s_i^n) > \frac{a}{n} Q_n.$$

Since, by A.1 $|f(x|s_i)/f(x|s_i') - 1| < \alpha|s_i - s_i^n|$, it follows that $|f(s_{-i}|s_i)/f(s_{-i}|s_i^n) - 1| < \alpha|s_i - s_i^n|$. Thus, there exists ε such that for any $s_i \in [s_i^n + \varepsilon, s_i^n + 2\varepsilon]$

$$E(q_i^n(s_i^n, s_{-i})|S_i = s_i) > \frac{a}{2n} Q_n.$$

In a *self-enforcing* mechanism based on Lemma 1,

$$E[q_i^n(s_i^n, S_{-i})v(s_i^n, X) - t_i^n(s_i^n, S_{-i})|s_i] \geq 0, \tag{A.11}$$

for any $s_i \in [s_i^n + \varepsilon, s_i^n + 2\varepsilon]$. Since the good has a private value component, and by A(II), we know that there exists $\tau > 0$ such that $v(s_i, x) - v(s_i^n, x) > \tau\varepsilon$ for any x. Thus,

$$E[q_i^n(s^n, s_{-i})\{v(s_i, X) - v(s^n, X)\}|s_i] \geq \tau\varepsilon \frac{a}{2n} Q_n \tag{A.12}$$

Since

$$\text{Sur}^n(s_i, s_i^n) = E[q_i^n(s_i^n, s_{-i})v(s_i, x) - t_i^n(s_i^n, s_{-i})|s_i],$$

(A.11) and (A.12) imply that for any $s_i \in [s_i^n + \varepsilon, s_i^n + 2\varepsilon]$

$$\text{Sur}^n(s_i, s_i^n) \geq \tau\varepsilon \frac{a}{2n} Q_n.$$

By incentive compatibility,

$$\text{Sur}^n(s_i, s_i) \geq \tau\varepsilon \frac{a}{2n} Q_n$$

This shows that conditional on getting a signal in $[s_i^n + \varepsilon, s_i^n + 2\varepsilon]$ any agent expects a surplus that is bounded away from zero (relative to the per-capita supply of objects Q^n/n). The positive density of $f(s_i)$ implies that there is a minimum positive probability that signals fall in $[s_i^n + \varepsilon, s_i^n + 2\varepsilon]$ regardless of the choice of s_i^n, and hence the claim follows. □

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