Online Algorithms and Option Pricing

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Abstract

In this work we show how to use efficient online trading algorithms to price the current value of financial instruments, such as an option. We derive both upper and lower bounds for pricing an option, using online trading algorithms. Our bounds depend on very minimal assumptions and are mainly derived assuming that there are *no arbitrage* opportunities.

1 Introduction

Options have been used from the time of Ancient Romans, Grecians, and Phoenicians for risk management. A call option is a contract between two parties (buyer and seller, or writer) that provides the buyer with insurance against appreciation in the price of a risky asset. It is used to hedge risk associated with financial assets (such as equities and currencies) as well as non-financial assets (commodities such as oil).

A (European) call option on a risky asset gives you the right but not the obligation to buy a risky asset on a pre-specified date, T, at a pre-specified price, K, referred to as the *strike price*. For example, a $T=$ 1-year call option on IBM with a strike price $K=\$100$ gives you the right but not the obligation to buy an IBM share from the writer of the option for a price of \$100. If we denote the value of the risky asset at time t by S_t then at time T the payoff of the call option is given by:

$$\max\{S_T - K, 0\}.$$

A fundamental question in finance is to determine the value of such an option today. This research question (which has many practical implications) has been a source of one of the main achievements in economics. Black and Scholes [3] studied this question in a path breaking paper that was later recognized by the 1997 Noble Prize. They show how one can replicate the payoff of an option using a dynamic trading strategy; based on this they provide an exact current valuation of the option. Their approach uses a no arbitrage condition and is based on the assumption that the logarithm of the stock price follows a certain continuous time version of a random walk[1]. They also assume that the stock and a risk free bond can be traded continuously.

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[1]More specifically they assume Brownian motion with a drift; for more details see for example [12]

This assumption on the stock price process is an important limitation of the Black-Scholes model. In practice share prices exhibit behavior that is not consistent with the simple random walk assumption made by the Black and Scholes model. Most importantly, trading is discrete and the price paths are discontinuous and include price jumps. Since Black and Scholes, there has been much research both in extending the results to alternative stock price processes, and in examining the empirical performance of the resulting valuation models. (We review briefly some of the related finance research in the Appendix.) The main goal of our work is to provide robust upper and lower bounds for European call options. The bounds we provide are robust in the sense that we do not restrict the form of the stock price process, we do not require continuous trading, and we allow for price jumps.

At this point it would be worthwhile to define our notation. We assume that the risky asset at time t is valued at $S_t = S_{t-1}(1+r_t)$, and that the returns r_t are bounded by some constant M, i.e., $|r_t| \leq M$. We call the sequence of r_t's the *price path*. Let $C(K,T)$ be the current price of an option with a strike price K that matures at time T, i.e., at time T the option payoff is $\max\{S_T - K, 0\}$. We also assume a risk free asset is available and it pays a constant interest rate r_f. For convenience, we normalize the interest rate to be 0 initially and show how to relax this assumption later.

We make two assumptions regarding the price path. Our main assumption bounds the *quadratic variation* of the risky asset, i.e., $\sum_t r_t^2$. Specifically for our upper bound on the option price we use an assumption of the form: [2]

$$\sum_{t=1}^{T} r_t^2 < Q.$$

We make an additional assumption regarding the *maximal single-period return* of the stock by restricting $|r_t| \leq M$. However this bound is weak and has a very mild effect on our price bounds for the option. (Black and Scholes effectively restrict $M = 0$ as they require continuous price paths.)

Unlike the Black and Scholes model, in our case we make no further assumptions on the underlying process, and allow for both price jumps and discreteness in trading. The advantages of our bounds is that they apply in a very adversarial setting, where the only restriction on the adversary is regarding the maximum quadratic variation and maximal single-period return. Another advantage is that the bounds are dependant on the quadratic variation in discrete time, which imply that if we make the trading less frequent our bounds do not deteriorate significantly.

When examining our bound it is important to note that without bounding the quadratic volatility there is little we can say about the value of the call option. As Merton [27] shows one can only say that the value of a call option cannot be higher than the current share price S_0 and cannot be negative or lower than the difference between the current share price and the strike price. Hence, the bound for the value of a call option is given by:

$$C(K,T) \in \left[\max\left\{0, S_0 - K\right\}, S_0\right].$$

The above bound is based on an arbitrage condition; if the price of the call option is outside this range then one can construct an arbitrage strategy that is guaranteed to make money. These bounds are tight as there does not exist an arbitrage for prices in this set if we do not put any restriction on the price path of the risky asset. In this work we demonstrate that it is possible to significantly improve on these bounds by assuming an upper bound on the quadratic variation and the maximal single-period return.

[2] The quadratic variation is closely related to the *volatility* parameter in the Black and Scholes model. In the Black-Scholes setting the quadratic variation converges to the stock's volatility when we observe the stock price at finer and finer time increments; hence, in practice quadratic variation is often used to estimate volatility.

Our approach is based on online algorithms and in particular the *best expert* problem [26, 5, 19, 1, 6] and regret minimization ideas [17, 18, 20, 25]. To illustrate our approach consider the following example. Suppose the current IBM share price is \$100 and the risk free interest rate is zero. Assume that we have an online trading algorithm such that if we start with \$100 then in time T our payoff will exceed $\max\{80, 0.8S_T\}$, where S_T denotes IBM share price at time T. (This can be viewed as a loss of not more than 20% compared to the best asset in hindsight, namely $\max\{100, S_T\}$.) By scaling our online trading algorithm we can conclude that starting with \$125 our algorithm would have a payoff that exceeds $\max\{100, S_T\}$. If we borrow \$100 to initiate our online algorithm, then after paying off our (zero interest) loan, the final payoff of our online algorithm would exceeds $\max\{0, S_T - 100\}$, which is identical to the payoff of a call option on IBM with a strike price of \$100. Thus, to avoid arbitrage, the value of the call option cannot exceed the upfront cost of \$25 of the online trading algorithm.

In the above example the quality of our bound is determined by the loss of our online trading algorithm relative to best static decision (which is either to buy the stock or not to buy it). A loss of 20% translated into a bound of \$25 and a better guarantee would translate into a lower loss and hence better (lower) upper bound. In our analysis we use online algorithms to minimize the loss given a bound on the quadratic variation of the risky asset.

As a first step in constructing our upper bound, we derive an online trading algorithm `generic` that has a relatively small loss. Our algorithm borrows ideas from the recent work of [6], however the analysis is adapted to the special case of a multiplicative model, which is widely used in finance. The `generic` online trading algorithm, assuming that the quadratic variation is bounded by Q, ends with value G_T, and has a guarantee of

$$\ln(G_T) \geq \max\{\ln(S_T) - \frac{1}{\eta}\ln\frac{1}{w} - (\eta - 1)Q \ , -\frac{1}{\eta}\ln\frac{1}{1-w}\} \ ,$$

where we normalize $S_0 = 1$, and both $w \in (0, 1)$ and $\eta \geq 1$ are parameters of the algorithm.

Given the `generic` online trading algorithm we derive the upper bound for the price of the call option. By varying the parameters of the algorithm, we are able to produce upper bounds for the entire range of possible strike prices. We also show how to incorporate non-zero interest rates in our model. Furthermore, we show that we can integrate other assumptions to get improved bounds, for example by assuming an upper or lower bound on the final risky asset price S_T. Finally, we discuss and compare our bounds to that of Black and Scholes.[3]

We complement our upper bound on the price of an option by introducing a lower bound. Specifically, we show that if the price path is guaranteed to have a quadratic variation of at least Q_{min}, then the value of the option must exceed this lower bound. In order to derive the lower bound we need to demonstrate an online trading strategy that would have a "guaranteed loss", regardless of the future outcomes. Based on such an online trading strategy and the no arbitrage assumption, we derive our lower bound. Note that our lower bound is not limited to a specific instance of prices but rather provides a no-arbitrage lower bound.

As mentioned earlier, our research builds on existing results in online learning and in particular *regret minimization*. The regret minimization problem has been extensively studied in the computational learning theory community and tight bounds have been derived on its behavior [26, 5, 19, 1]. In a nutshell, one can reach almost the gain of the best action, assuming that we are allowed to output a linear combination of the actions (or alternatively allowed to use randomization). More precisely, if the gain of the best action is G, then there exists an online algorithm whose gain is at least $G - O(\sqrt{G \log N} + \log N)$.

[3] One of the main advantage of our bounds is that since they depend on the quadratic variation, they allow for a very natural comparison to the standard literature in finance which uses volatility.

Common to much of the previous computer science work on online trading algorithms is the goal of *beating the market*. To do so requires designing an online algorithm that can have a guarantee of performing very well in the market, even under adversarial conditions. Much of the Finance literature assumes that assets are priced fairly, and at the very least assumes that there are no arbitrage opportunities, which implies that *no* online algorithm can "beat the market". A significant conceptual contribution of our work, to the computer science view point, is that efficient online algorithms can play an important role even if we assume *no arbitrage* in the market prices. Our goal is much more modest, we try to price fairly a given financial instrument, and to this end we do need to design *efficient* online algorithms. Another observation is regarding the *static adversary* in the online model. Essentially, an option is an "insurance" contract to get the performance of a static adversary, which in hindsight selects between the risky and risk free asset. While in much of the online competitive literature a static adversary is viewed as an extremely limited adversary, in the finance setting it has both a very natural interpretation and very significant applications.

We briefly outline the main research direction in online algorithms regarding financial problems. The *universal portfolio*, proposed by [9] and latter studied in [4, 24, 32, 21, 10, 11, 33], has been one of the well studied finance problem in the information theory and computer science literature. In this setting the aim is to devise an online trading algorithm that is competitive against *any* constant rebalancing portfolio. Another finance-motivated problem that has received considerable attention is the *one-way trading* problem, first studied in [14], and latter in [7, 13]. In this problem one needs, for example, to change a fixed amount of Yen to Dollars. The aim is to compare well with *any* trading strategy, even one that knows the future prices and trades at the best price. The results derive competitive online trading algorithms, and are highly related to search problems. A related finance motivated problem is Volume Weighted Average Price (VWAP) trading [22]. Extensions of the one way trading to *two way trading* appear in [30, 7, 23], where various limitations are assumed regarding the adversary. (Part of the assumptions are aimed at either bounding the offline benefit or guaranteeing that the online algorithm will not lose money.)

2 Model

We consider a discrete-time finite-horizon model in which time is denote by $t \in \{0, 1, \ldots, T\}$. There is a risky asset (e.g., stock) whose value (price) at time t is given by S_t. We normalize the initial value to one, $S_0 = 1$, and assume that the asset does not pay any dividends. We denote by r_t the return between $t-1$ and t so that $S_t = S_{t-1}(1 + r_t)$. We call $r_1, \ldots, r_T$ the *price path* .

In addition to the risky asset we have a risk free asset (e.g., bond). We denote the price of the risk free asset at time t by B_t, where $B_0 = 1$. We assume that the risk free asset carries a fixed interest r_f, i.e., $B_{t+1} = B_t(1 + r_f) = (1 + r_f)^t$. Unless otherwise stated, we assume that the risk free rate is zero, i.e., $r_f = 0$, which implies that $B_t = 1$ for all t.

A *financial security* X has for each price path $r_1, ..., r_T$ a payoff of $X(r_1, ..., r_T)$. For example, an option can be described as a financial security.

An *online trading algorithm* starting with $\$c$ in cash, has initial value $G_0 = c$. At each period it distributes its current value G_t, between the assets, investing a fraction x_t in the risky asset and $1 - x_t$ in the risk free asset. Since we assume zero interest rate, at time $t + 1$ its value is $G_{t+1} = (x_t G_t)(1+r_t) + (1-x_t)G_t = G_t(1+x_t r_t)$. Its final value is G_T. We refer to a *fixed portfolio* when the online trading algorithm sets its investments at time $t = 0$, and does not trade anymore. Formally, it implies that $x_{t+1} = x_t(1 + r_t)/(1 + x_t r_t)$, since the value of the risky asset changes.

Let $C(K, T)$ be the value, at time $t = 0$, of an option whose strike price is K that matures at time T. This is the present value (at $t = 0$) of a time T call option with strike price K. At time T

the payoff of the call option is given by $\max\{0, S_T - K\}$.

A price path $r_1, \ldots, r_T$ is a (Q, M) *price path* if $\sum_{t=1}^{T} r_t^2 < Q$ and $|r_t| < M$. In the case of a risky asset and a risk free asset, a (Q, M) price path means that both the price paths, of the risky asset and the risk free asset, are (Q, M) price path. (For the risk free asset this means that $r_f^2 T < Q$ and $r_f < M$, which holds trivially for $r_f = 0$.) We call Q the *maximum realized quadratic variation* of the risky asset and M the *maximal single period return.*

No Arbitrage Assumption: We assume that there is *no arbitrage* in the prices. Namely, for any two online trading algorithms (or financial securities) A_1 and A_2, that start with cash $\$c_1$ and $\$c_2$, if for any price path the future payoff of A_1 is always at least that of A_2, then $c_1 \geq c_2$. (If this were not true and $c_1 < c_2$, assuming that one can sell short assets (and strategies), there would be an arbitrage opportunity: Investing in A_1 and shorting A_2 would lead to a time 0 gain of $c_2 - c_1$ without the possibility of loss in the future.)

We will use the no arbitrage assumption on a restricted set of price paths, such as (Q, M) price paths. This implies that any price path which is not in the set is impossible, and the prices reflect this knowledge.

3 An online trading strategy: `generic`

In this section we introduce an online trading algorithm `generic`. The online algorithm will trade using N different assets, and its goal is to have its value approximate the value of the *best* asset.[4] Later we shall see how a simple application of `generic` indeed yields the desired upper bound on the price of the option.

Notation: We denote by $V_{i,t}$ the price of asset i at time t. We normalize the initial value of each asset to be one, i.e., $V_{i,0} = 1$. The value at time t satisfies $V_{i,t} = V_{i,t-1}(1 + r_{i,t})$, where $r_{i,t} \in [-M, +M]$ is the immediate return of asset i at time t.

Algorithm `generic`: The online trading algorithm `generic` maintains weights, $\{w_{i,t}\}$. Initially $\sum_i w_{i,0} = 1$, where the exact setting is a parameter of the algorithm. The algorithm uses the update rule $w_{i,t+1} = w_{i,t}(1 + \eta r_{i,t})$, for some parameter $\eta \geq 0$. At time t the algorithm forms a portfolio where the fraction of investment in asset i is $x_{i,t} = w_{i,t}/W_t$ where $W_t = \sum_i w_{i,t}$.

Algorithm's Value: The value of the assets that the online trading algorithm `generic` holds at time t is denoted by G_t. Initially, $G_0 = 1$, and $G_t = G_{t-1}(1 + r_{G,t})$, where the immediate return on our portfolio at time t is $r_{G,t} = \sum_{i=1}^{N} x_{i,t} r_{i,t}$. Another way of describing the evolution of the value is: $G_t = \sum_{i=1}^{N} (x_{i,t} G_{t-1})(1 + r_{i,t}) = G_{t-1}(1 + r_{G,t})$.

The following theorem, whose proof appears in the Appendix, summarizes the performance of our online algorithm.

Theorem 1 *The online trading algorithm* `generic`*, given parameters:* $\eta \in \left[1, \frac{1}{M}\left(1 - \frac{1}{2(1-M)}\right)\right]$*, and* $\{w_{i,0}\}$*, where* $\sum_i w_{i,0} = 1$*, guarantees that for any asset* i*,*

$$\ln(G_T) \geq \ln(V_{i,T}) - \frac{1}{\eta} \ln \frac{1}{w_{i,0}} - (\eta - 1) Q_i \ ,$$

where $Q_i = \sum_{t=1}^{T} r_{i,t}^2$*, and* $|r_{i,t}| < M < 1 - \sqrt{2}/2 \approx 0.3$*.*

Our option pricing results will rely on an application of the above theorem to a setting with two assets: a risky asset and a risky free asset. With a slight abuse of notation we let w_0 denote the

[4] The algorithm and analysis is in the spirit of [6], however, note that the model there is additive while the model here is multiplicative.

amount invested in the risky asset and assume that we invest $1 - w_0$ in the risk free asset. Since we assume a zero interest rate we have $Q_f = 0$ for the risk free asset and conclude that:

Corollary 2 *The online trading algorithm* **generic***, given parameters:* $\eta \in \left[1, \frac{1}{M}\left(1 - \frac{1}{2(1-M)}\right)\right]$*, and* $w_0 \in (0,1)$*, when applied to a risky asset and a risk free asset, guarantees that*

$$\ln(G_T) \geq \max\left\{\ln(S_T) - \frac{1}{\eta}\ln\frac{1}{w_0} - (\eta - 1)Q \ , -\frac{1}{\eta}\ln\frac{1}{1-w_0}\right\} \ ,$$

where $Q = \sum_{t=1}^{T} r_t^2$*, and* $|r_t| < M < 1 - \sqrt{2}/2 \approx 0.3$.

4 Robust Option Pricing Upper bound

We define our upper bounds using an online trading algorithm. Our bounds are based on an online trading algorithm whose payoff always exceed the option payoff, provided we have a (Q, M) price path. The fact that this derives an upper bound on the price of an option would follow from the no arbitrage assumption.

Definition 3 *We say that* $c = C^u(K, Q, M, T)$ *is an upper bound if there exists an online trading algorithm that starts with \$c and for all possible* (Q, M) *price paths its final payoff,* G_T*, satisfies:* $G_T \geq \max\{0, S_T - K\}$.

As we shall discuss next, we actually examine an equivalent guarantee as follows:

Definition 4 *An online trading algorithm has an* (α, β) *guarantee, if for any* (Q, M) *price path its final payoff,* G_T*, satisfies* $G_T \geq \max\{\alpha, \beta S_T\}$.

To gain some intuition it is better to first examine a very simple trading strategy. Suppose we decide to use a buy and hold strategy in which we invest a fraction β in the risky asset and $\alpha = 1 - \beta$ in the risk free asset. The future payoff of this fixed portfolio, G_T, is

$$G_T = \alpha + \beta S_T \geq \max\{\alpha, \beta S_T\}$$

This implies that we implemented an (α, β) guarantee for $\beta = 1 - \alpha$. Compare the above to the payoff of a fixed portfolio of β call options each with a strike price of $K = \frac{\alpha}{\beta}$ combined with α invested in the risk free asset. Such a fixed portfolio yields at time T a payoff of exactly $H_T = \alpha + \beta\max\{0, S_T - (\alpha/\beta)\} = \max\{\alpha, \beta S_T\}$. By definition, the current price of this fixed portfolio is $\beta C(\frac{\alpha}{\beta}, T) + \alpha$. Since $G_T \geq H_T$, by the no arbitrage assumption, we have,

$$\beta C(\frac{\alpha}{\beta}, T) + \alpha \leq 1 \qquad \Rightarrow \qquad C(\frac{\alpha}{\beta}, T) \leq \frac{1-\alpha}{\beta} = 1 = S_0$$

As mentioned before, S_0 is a simple known upper bound on the option price. Our goal is to construct online trading algorithm that starts with \$1 and yields a future payoff that exceeds:

$$\max\{\alpha, \beta S_T\},$$

for some $\alpha + \beta > 1$. Such an algorithm yields a non trivial bound, as stated in the following claim,

Claim 5 *Assume that all price paths are* (Q, M) *price paths. An online trading algorithm with an* (α, β) *guarantee ensures that for a call option with strike price* $K = \frac{\alpha}{\beta}$*,* $C^u(K, Q, M, T) \leq \frac{1-\alpha}{\beta} = \frac{1}{\beta} - K$.

4.1 Robust upper bounds

We will use the `generic` online trading algorithm to generate our upper bounds for the value of the options. The main feature of the `generic` algorithm is that it tries to match the best of the underlying assets, which intuitively, is what we need to generate our bound. From Corollary 2 we have,

$$G_T \geq \max\{\alpha, \beta S_T\}$$

where

$$\alpha(w_0, \eta) = (1 - w_0)^{1/\eta} \qquad \text{and} \qquad \beta(w_0, \eta) = w_0^{1/\eta} e^{-(\eta-1)Q}.$$

Now consider the bound for a given strike price K. We obtain this by solving:

$$\beta^*(K) = \max_{\eta, w_0} \beta(w_0, \eta) \qquad \text{s.t.}$$
$$\frac{\alpha(w_0,\eta)}{\beta(w_0,\eta)} = K \qquad \text{and} \qquad \eta \in \left[1, \frac{1}{M}\left(1 - \frac{1}{2(1-M)}\right)\right]$$

One can simplify this problem by using $\frac{\alpha(w_0,\eta)}{\beta(w_0,\eta)} = K$ to solve for w_0:

$$w_0(\eta, K) = \frac{1}{1 + K^{\eta} e^{-\eta(\eta-1)Q}}$$

Hence, we need to solve the following maximation,

$$\beta^*(K) = \max_{\eta} w_0(\eta, K)^{1/\eta} e^{-(\eta-1)Q} \qquad \text{s.t.} \qquad \eta \in \left[1, \frac{1}{M}\left(1 - \frac{1}{2(1-M)}\right)\right]$$

Let $\beta^*(K)$ be the solution to the above optimization, our bound is then given by

$$C(K,T) \leq C^u(K,Q,M,T) \leq \frac{1}{\beta^*(K)} - K$$

Special case: $K = S_0 = 1$: In order to gain better intuition regarding our upper bound we derived a crude estimate of it, in the special case of $K = 1$, also referred to as an "at the money" call option. For the case $K = 1$ the maximization simplifies to $\beta^*(1) = \max_{\eta} \left(1 + e^{\eta(\eta-1)Q}\right)^{-1/\eta}$. Consider a specific setting of η, where $\eta^* = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{Q}}$. This implies that $\eta^*(\eta^* - 1)Q = 1$ and $\beta^*(1) \geq (1+e)^{-1/\eta^*}$. We can approximate our upper bound as follows,

$$C(1,T) \leq C^u(1,Q,M,T) \leq (1+e)^{1/\eta^*} - 1 \leq 1 + \frac{e}{\eta^*} - 1 = \frac{e}{\eta^*} = \sqrt{Q}\left[\frac{e}{2}(\sqrt{4+Q} - \sqrt{Q})\right] \leq e\sqrt{Q}$$

where we used the inequality $(1+a)^{1/n} \leq 1 + a/n$ for $n \geq 1$, and the fact that $\eta^* \geq 1$. In order to keep η^* in the right range, as a function of M, we need to add an assumption such as $M < \min\{1/4, 1/(3\eta^*)\}$. While the optimization is guaranteed to yield an upper bound less than the trivial upper bound $S_0 = 1$, to gain intuition we have made a number of approximations above to provide a crude estimate of the upper bound. The above result is, of course, only interesting for $Q < e^{-2}$.

Comparison to Black and Scholes: One question that might arise is whether Black and Scholes price already an upper bound if one fixes the quadratic variation. One can show simple examples (even with $T = 2$) where the Black and Scholes price is not an upper bound. In Figure 1 we show a comparison of our bound to that of Black and Scholes (where the our maximum realized quadratic variation corresponds to the volatility parameter). As we can see from the graph, our upper bound is similar in shape to the Black and Scholes pricing.

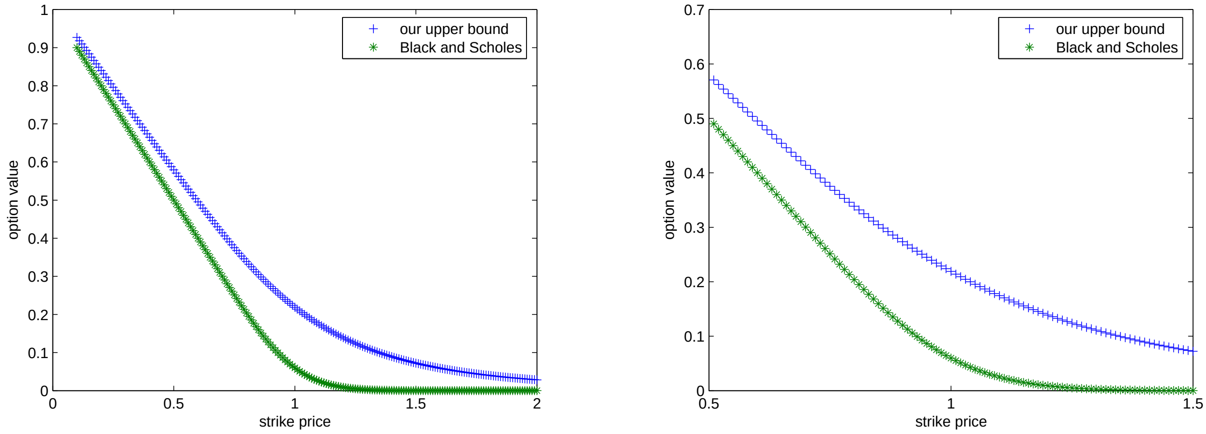

Figure 1: Graph comparing our upper bound to Black and Scholes using $Q = (0.15)^2$, and $M = 0.1$.

4.2 Extensions

In this section we outline a few interesting extension of the basic model, and sketch how to handle them in our framework.

Positive interest rate: We have assumed so far that the interest rate is zero, we now show how to handle positive interest rates; specifically we assume that the risk free rate is given by $r_f > 0$. There are several possible ways to handle this case and we sketch only the most straightforward one. We can consider the risk free asset (e.g., bond) as a second asset in `generic` with a fixed price change of r_f and $Q_f = r_f^2 T$. This implies that now the value of α is,

$$\alpha\left(w_0, \eta\right) = \left(1 - w_0\right)^{1/\eta} e^{-(\eta-1)Q_f} \left(1 + r_f\right)^T,$$

and now we need to make the optimization with the new function of $\alpha(w_0, \eta)$.

Maximum and minimum price: One can add another reasonable assumption about the final price of the risky asset, i.e., S_T. We can add an assumption that $S_T \in [S_{min}, S_{max}]$. A careful look at the proof of Theorem 1 for the `generic` algorithm reveals that we can take such information into account when generating our bound. We can show, using Lemma 14, that

$$S_T^\eta \geq w_T \geq S_T^\eta e^{-\eta(\eta-1)Q},$$

where w_T is the final weight of the risky asset and $Q = \sum_t r_t^2$. Using this we can modify Corollary 2, and show that for $r_f = 0$ we have,

$$\ln(G_T) \geq \max\left\{ \ln(S_T) - \frac{1}{\eta}\ln\frac{1}{w_0} - (\eta - 1)Q + \frac{1 - w_0}{S_{max}^\eta + 1 - w_0}\,, \right.$$
$$\left. -\frac{1}{\eta}\ln\frac{1}{1 - w_0 + w_0 S_{min}^\eta} + \frac{S_{min}^\eta e^{-\eta(\eta-1)Q}}{S_{min}^\eta e^{-\eta(\eta-1)Q} + 1 - w_0} \right\}.$$

As expected, the additional assumption can only improve our upper bound, and for $S_{min} = 0$ and $S_{max} = \infty$ we retrieve the previous bound. In fact, we can slightly improve the previous bounds by noting that given a bound on the realized quadratic variation Q, or maximal single period return M, we can bound the price of the risky asset. However the improvements are very small.

5 A robust lower bound

While the focus of this paper is on upper bounds we can derive a lower bound as a function of the minimum realized volatility. It is rather straightforward to derive a lower bound in a specific scenario with a given volatility, an example is the Black and Scholes model that provide such a bound since it derives the exact value of the option given a specific stochastic model. In this section we derive much stronger lower bounds. We show that, assuming that every possible scenario that has at least the minimum realized volatility, we can derive a lower bound for the price of an option.

Definition 6 *We say that $c = C^l(K, Q_{min}, M, T)$ is a lower bound for $C(K, T)$, if there exists an online trading algorithm that starts with \$c and its final payoff, G_T, satisfies $G_T \leq \max\{0, S_T - K\}$, for all possible price paths for the risky asset that satisfy $\sum_{t=1}^{T} r_t^2 \geq Q_{min}$ and maximal single period return of M.*

Again we construct a strategy that provides an alternative but equivalent guarantee. Suppose that we construct a trading strategy that starts with \$1 and guarantees a future payoff that is *always less* than $\max\{\alpha, \beta S_T\}$. In such a case we can derive the following lower bound.

Claim 7 *Assume that there exists an online trading algorithm that starts with* \$1 *and its final payoff is at most* $\max\{\alpha, \beta S_T\}$. *Then* $C(\frac{\alpha}{\beta}, T) \geq C^l(\frac{\alpha}{\beta}, Q_{min}, M, T) \geq \frac{1-\alpha}{\beta}$

Proof: Consider two online trading algorithms. Algorithm A_1, as stated in the claim, is an online trading algorithm which starts with \$1, and whose future payoff is bounded by $\max\{\alpha, \beta S_T\}$. Algorithm A_2 buys a fixed portfolio of β call options each with a strike price $K = \frac{\alpha}{\beta}$ plus \$$\alpha$ risk free asset. The future payoff of A_2 is $\alpha + \beta \cdot \max\{0, S_T - (\alpha/\beta)\} = \max\{\alpha, \beta S_T\}$. Since the payoffs of A_2 dominates the payoffs of A_1, by the no arbitrage condition we have that $\beta C(\frac{\alpha}{\beta}) + \alpha \geq 1$, which proves the claim. □

Our bound is interesting when it improves over the standard bound, $\max\left\{1 - \frac{\alpha}{\beta}, 0\right\}$; that is:

$$\frac{1-\alpha}{\beta} > \max\left\{1 - \frac{\alpha}{\beta}, 0\right\},$$

which holds when both $\alpha, \beta \in (0,1)$. Note that one can guarantee $\alpha = 0$ and $\beta = 1$ by simply only investing in the risky asset. It is our ability to guarantee that both $\alpha < 1$ and $\beta < 1$ simultaneously, that will generate interesting bounds. Hence, the objective is very different from the standard objective in online algorithms. We are interested in guaranteeing a certain "loss" as opposed to guaranteeing a maximal gain.

5.1 Combining online trading strategies

It would be helpful to develop an online trading algorithm, where we have a certain guarantee conditional on a certain event. Formally we denote by A a certain event that hold for the price sequence and by A^c its complement. The following definition formalizes what are the properties we like to have when we condition on an event.

Definition 8 *For a given $\alpha_X, \beta_X \in (0,1)$ and an event A, an online trading algorithm X that starts with* \$1 *is said to be* bounded by (α_X, β_X) *on A if (i) X yields a payoff that is always smaller than* $\max\{1, S_T\}$, *and (ii) when A holds X yields a payoff that is smaller than* $\max\{\alpha_X, \beta_X S_T\}$.

Suppose that we have two trading strategies that work on events that are complements. We can combine two online trading algorithms, each is bounded on a complementing event, to derive an online trading algorithm which is bounded always.

Claim 9 *Assume online trading algorithm X is bounded by (α_X, β_X) on A and an online trading algorithm Y that is bounded by (α_Y, β_Y) on A^c. We can generate an online trading algorithm $Z(\gamma)$ that is always bounded by $\max\{\alpha_Z, \beta_Z S_T\}$, where*

$$\alpha_Z = \max\{\gamma\alpha_X + (1-\gamma)\ ,\ \gamma + (1-\gamma)\alpha_Y\}$$

$$\beta_Z = \max\{\gamma\beta_X + (1-\gamma)\ ,\ \gamma + (1-\gamma)\beta_Y\}$$

Proof: We construct a combined online trading algorithm $Z(\gamma)$ that runs X starting with $\$\gamma$ and runs Y with $\$\,(1-\gamma)$. If event A holds, then the payoff of $Z(\gamma)$ is at most $\gamma\max\{\alpha_X, \beta_X S_T\} + (1-\gamma)\max\{1, S_T\}$. This implies that $Z(\gamma)$ payoff is bounded by $\max\{\alpha_1, \beta_2 S_T\}$, where $\alpha_1 = \gamma\alpha_X + (1-\gamma)$ and $\beta_1 = \gamma\beta_X + (1-\gamma)$. A similar bound holds for the case that A^c holds, using the guarantee for Y. □

5.2 Bounded trading strategies

We consider now an online trading algorithm which is a variation of our online trading algorithm `generic`, which we call `square-momentum`. At time t `square-momentum` invests in the risky asset a fraction of $x_t = \frac{S_t^2}{1+S_t^2}$ and in the risk free asset a fraction of $1 - x_t = \frac{1}{1+S_t^2}$ in the risk free asset. Define the event $A_{\rho_1,\rho_2} = \{\forall t : S_t \in [1-\rho_1, 1+\rho_2]\}$, where we will specify the parameters ρ_1 and ρ_2 latter. The following lemma derives the performance of the `square-momentum` algorithm in case the assumption holds. (The proof appears in the Appendix.)

Lemma 10 *Assume that the maximal single period return is M. The `square-momentum` trading strategy guarantees that: (1) For every run $G_T \le \max\{1, S_T\}$, (2) if event A_{ρ_1,ρ_2} holds then $G_T \le \max\{\alpha, \beta S_T\}$, for $\alpha = \beta = e^{-h(\rho,M)Q_{min}}$, where $h(\rho, M) = \frac{(1-\rho)^2}{2(1+(1-\rho)^2)^2(1+M)^2} \le 1/8$ and $\rho = \max\{\rho_1, \rho_2\}$.*

We now define a `trade-once` online trading algorithm. Initially it starts with $x_1 = 1/2$ and does no trade as long as $S_t \in [1-\rho_1, 1+\rho_2]$. Once the condition is first violated it trade (and only once). If $S_t > 1+\rho_2$ it changes to $x_t = 1$ (buys only the risky asset) and does not trade any more. If $S_t < 1-\rho_1$ it changes to $x_t = 0$ (buys only the risk free asset) and does not trade any more.

Claim 11 *The `trade-once` trading strategy guarantees that: (1) For every run $G_T \le \max\{1, S_T\}$, (2) if $A^c_{\rho_1,\rho_2}$ holds then $G_T \le \max\{\alpha, \beta S_T\}$, for $\alpha = 1 - \frac{1}{2}\rho_1$ and $\beta = \frac{\rho_2+2}{2(1+\rho_2)}$.*

Proof: In case `trade-once` does not trade, we are left with a payoff $(1+S_T)/2$ which is bounded by $\max\{1, S_T\}$. In case `trade-once` does trade, our payoff is bounded by $\frac{\rho_2+2}{2(1+\rho_2)} S_T \le S_T$ if the stock gains in value, and by $1 - \frac{1}{2}\rho_1 \le 1$ if the stock loses in value. □

A simple corollary, when $\rho_1 = \rho_2 = \rho$ is the following.

Corollary 12 *The `trade-once` trading strategy, with $\rho_1 = \rho_2 = \rho \in (0,1)$, is bounded by (α, α) on A^c_ρ, for $\alpha = 1 - \frac{1}{2}\rho$.*

Proof: Since $\frac{\rho+2}{2(1+\rho)} > 1 - \frac{1}{2}\rho$ for $\rho \in (0,1)$ we conclude that the `trade-once` online trading algorithm is bounded by $\left(1 - \frac{1}{2}\rho, 1 - \frac{1}{2}\rho\right)$ on A^c_ρ. □

We can now combine Lemma 10 and Corollary 12, setting $\gamma = \frac{\rho/2}{1+\rho/2-e^{-h(\rho,M)Q_{min}}}$, and conclude:

Corollary 13 *Assume that the maximal single period return is M. There exists an online trading algorithm that is bounded by (α, α), where where $\alpha = \frac{1-(1-\rho/2)e^{-h(\rho,M)Q_{min}}}{1+(\rho/2)-e^{-h(\rho,M)Q_{min}}}$ and $\rho \in (0,1)$.*

An implication of the above corollary is the following lower bound on a call at the money (i.e., $K = S_0 = 1$),

$$C^l(1, Q_{min}, M, T) > \frac{1-\alpha}{\alpha} \geq \frac{(\rho/2)(1 - e^{-h(\rho,M)Q_{min}})}{1-(1-\rho/2)e^{-h(\rho,M)Q_{min}}}$$

To gain better intuition, note that when $Q_{min} \approx 0$ our lower bound is almost trivial (approximately 0). As Q_{min} increases (for a fix ρ and M) our bound increases.

Again, we would like to derive a crude approximation of our lower bound, in order to see the dependency on the parameter Q_{min}. Since the function $f(x) = a(1-x)/(1-(1-a)x)$, for $a > 0$, is monotone decreasing in $x \in (0,1)$, and using the fact that $e^{-z} \geq 1 - z$, we have,

$$C^l(1, Q_{min}, M, T) > \frac{(\rho/2)(1 - e^{-h(\rho,M)Q_{min}})}{1-(1-\rho/2)e^{-h(\rho,M)Q_{min}}} \geq \frac{(\rho/2)h(\rho,M)Q_{min}}{1-(1-\rho/2)(1-h(\rho,M)Q_{min})} \geq \frac{\rho h(\rho,M)}{1+\rho/2} Q_{min},$$

assuming $h(\rho, M)Q_{min} < 0.5$. For example, setting $\rho = 0.5$ and $M = 0.25$, this crude approximation gives a lower bound larger than $0.1Q_{min}$.

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A Literature review: Finance

While the Black-Scholes is one of the most useful formulas in economics, several empirical regularities have been observed. In recent years there has been an extensive empirical research that tests whether this formula holds in the data. In general the formula seems to generate prices that are too low. This effect is more pronounced for call option whose strike price, K, is low. This effect is often referred to as the 'volatility smile'. As a response to these findings, there has been active research (e.g., [29], [16] and [15]) trying to modify the Black and Scholes setting. These papers typically examine different stochastic processes than assumed by Black and Scholes. The modifications include jump processes and stochastic volatility models. The result of our study will complement this analysis. Rather than focusing on a specific formulation for the stochastic process we rely on a generic trading strategy that works with any evolution for the risky asset as long as it satisfies some requirements regarding the quadratic variation.

As a result of both academic and practical interest there are several papers that study what restrictions one can impose on the price of options. These papers are similar in spirit to our work as the goal is to provide a robust bound by relaxing the specific assumption made by Black and Scholes. Mykland [28] considers a stochastic process that is more general than what Black and Scholes assume: $dS_t = \sigma_t S_t dWt + \mu_t S_t dt$, where the volatility, σ_t, is allowed to be stochastic. In this case the market need not be complete[5] and we might be unable to replicate an option payoff. Still he shows that one can use the Black-Scholes price as an upper bound if we take the volatility parameter to be the upper bound over all realizations of the average stochastic volatility. The reason for this can be traced to Merton's argument [27] that if volatility is a known function of time, the Black and Scholes formula holds using the average volatility. While such bounds generalize Black and Scholes in a significant way they still impose significant restriction on the stochastic process. For example, the price path is assumed to be continuous so the stochastic price has no jumps; such jumps were shown to be empirically important [29, 16, 15]. For example, the Merton observation fails in a discrete time version of Black and Scholes; the ability to trade continuously is critical. Still the fact that [28], similar to our methodology, relies on an upper bound over the quadratic

[5] A complete market is one in which the existing assets allow all possible gambles on future outcomes.

volatility, can dramatically improve the bounds compared to the case in which one assumes an upper bound over the instantaneous volatility (see for example [31]).

An alternative approach to that taken here is developed by [2, 8], who strengthen the no-arbitrage condition by using an equilibrium argument. Specifically they assume bounds for the risk-reward ratio that should be achievable in the market. Based on these bounds and existing market prices, they can then determine upper and the lower bounds for new securities that may be introduced into the market.

B Proofs from Section 3

We first establish the following technical lemma.

Lemma 14 *Assuming that* $M \in (0,1)$, $r > -M$, *and* $\eta \in \left[1, \frac{1}{M}\left(1 - \frac{1}{2(1-M)}\right)\right]$ *then:*

$$\eta \ln(1+r) \geq \ln(1+\eta r) \geq \eta \ln(1+r) - \eta(\eta-1)r^2$$

Proof: For the first inequality define a function

$$f_1(r) = \eta \ln(1+r) - \ln(1+\eta r)$$

We have $f_1(0) = 0$, and

$$f_1'(r) = \frac{\eta}{1+r} - \frac{\eta}{1+\eta r} = \frac{\eta(\eta-1)r}{(1+r)(1+\eta r)}$$

Hence, for $r > 0$ then $f_1'(r) > 0$ and for $r < 0$ we have $f_1'(r) < 0$. Therefore 0 is a minimum point of f_1.

For the second inequality we have:

$$f_2(r) = \ln(1+\eta r) - \eta \ln(1+r) + \eta(\eta-1)r^2$$

Again, $f_2(0) = 0$ and

$$f_2'(r) = \frac{\eta}{1+\eta r} - \frac{\eta}{1+r} + 2\eta(\eta-1)r = \eta(\eta-1)r\left(2 - \frac{1}{(1+r)(1+\eta r)}\right)$$

We use a similar argument as before and claim that for $r > 0$ then $f_2'(r) > 0$ and for $r < 0$ we have $f_2'(r) < 0$. To show this we only need to verify that:

$$(1+r)(1+\eta r) \geq \frac{1}{2}$$

For $r > 0$ this clearly holds so we focus on $r < 0$. In this case, since the minimum of the expression is when $r = -M$, it is sufficient to guarantee that $(1-M)(1-\eta M) \geq 1/2$. Solving for η we get,

$$\eta \leq \frac{1/2 - M}{M(1-M)} = \frac{1}{M}\left(1 - \frac{1}{2(1-M)}\right),$$

and in addition we need that $M < 1$. $\square$

We now can prove the Theorem.

Proof of Theorem 1: For each $i = 1, \ldots, N$ we get

$$\begin{aligned}
\ln \frac{W_{T+1}}{W_1} \geq \ln \frac{w_{i,T+1}}{W1} &= \ln w_{i,0} + \ln \prod_{t=1}^{T} (1 + \eta r_{i,t}) \\
&= \ln w_{i,0} + \sum_{t=1}^{T} \ln(1 + \eta r_{i,t}) \\
&\geq \ln w_{i,0} + \sum_{t=1}^{T} \left(\eta \ln(1 + r_{i,t}) - \eta\, (\eta - 1)\, r_{i,t}^2 \right) \\
&= \ln w_{i,0} + \eta \ln(V_i) - \eta\, (\eta - 1)\, Q_i
\end{aligned}$$

where $V_{i,T}$ is the value of asset i at time T, and $Q_i = \sum_{t=1}^{T} r_{i,t}^2$.

On the other hand, using $\ln(1 + \eta z) \leq \eta \ln(1 + z)$,

$$\begin{aligned}
\ln \frac{W_{T+1}}{W_1} &= \sum_{t=1}^{T} \ln \frac{W_{t+1}}{W_t} \\
&= \sum_{t=1}^{T} \ln \sum_{i=1}^{N} (1 + \eta r_{i,t}) x_{i,t} \\
&= \sum_{t=1}^{T} \ln \left(1 + \eta \sum_{i=1}^{N} r_{i,t} x_{i,t} \right) \\
&= \sum_{t=1}^{T} \ln \left(1 + \eta r_{G,t}\right) \\
&\leq \sum_{t=1}^{T} \eta \ln \left(1 + r_{G,t}\right) \\
&= \eta \ln(G_T)\ .
\end{aligned}$$

Combining the two inequalities and dividing by $\eta \geq 1$, we get

$$\ln(G_T) \geq \frac{\ln w_{i,0}}{\eta} + \ln(V_{i,T}) - (\eta - 1) Q_i$$

□

C Proofs from Section 5

We first establish the following technical lemma.

Lemma 15 *For any $A, B \geq 0$ we have*

$$\ln(A + B) \geq \ln(A) + \frac{B}{A + B}$$

Proof: First note that $\ln(1-x) \leq -x$, for $x \in (0,1)$. Let $x = B/(A+B)$, then,

$$\ln(A) - \ln(A+B) = \ln(\frac{A}{A+B}) = \ln(1 - \frac{B}{A+B}) \leq -\frac{B}{A+B}$$

and the claim follows. □

Proof of Lemma 10: Consider the ratio between the initial and final weights, and recall that $r_{G,t} = x_t r_t$. We have,

$$\begin{aligned}
\ln\left(\frac{1+S_T^2}{2}\right) &= \sum_t \ln \frac{1+S_{t+1}^2}{1+S_t^2} \\
&= \sum_t \ln\left((1-x_t) + x_t\,(1+r_t)^2\right) \\
&= \sum_t \ln\left(1 + 2x_t r_t + x_t r_t^2\right) \\
&= \sum_t \ln\left(1 + 2r_{G,t} + x_t r_t^2\right) \\
&= \sum_t \ln\left((1+r_{G,t})^2 + x_t(1-x_t)r_t^2\right)
\end{aligned}$$

The fact that $x_t < 1$ implies that:

$$(1+r_{G,t})^2 + x_t(1-x_t)r_t^2 = 1 + 2x_t r_t + x_t r_t^2 < (1+r_t)^2.$$

Based on Lemma 15, we have:

$$\sum_t \ln\left((1+r_{G,t})^2 + x_t(1-x_t)r_t^2\right) \geq \sum_t 2\ln(1+r_{G,t}) + \frac{x_t(1-x_t)r_t^2}{(1+r_t)^2} \geq 2\ln(G_T) + \sum_t \frac{x_t(1-x_t)r_t^2}{(1+r_t)^2}$$

Since

$$\ln\left(\frac{1+S_T^2}{2}\right) \leq \ln\left(\max\{S_T, 1\}^2\right) = 2\ln\left(\max\{S_T,1\}\right)$$

we conclude that

$$\ln G_T \leq \ln\left(\max\{S_T,1\}\right) - \sum_t \frac{x_t(1-x_t)r_t^2}{2(1+r_t)^2}$$

This implies that, $G_T \leq \max\{S_T, 1\}$ and concludes the first part of the theorem.

For the second part of the theorem, we like to bound the additional term when the event A_{ρ_1,ρ_2} holds. Let $X(1-X)$ be some lower bound for $x_t(1-x_t)$, that we derive using the assumption for every t, A_{ρ_1,ρ_2} holds. Specifically, if A_{ρ_1,ρ_2} holds, i.e., $\forall t : S_t \in [1-\rho_1, 1+\rho_2]$, then for any t we have that $x_t \in \left[\frac{(1-\rho_1)^2}{1+(1-\rho_1)^2}, \frac{(1+\rho_2)^2}{1+(1+\rho_2)^2}\right]$. This implies that we can set

$$X = \frac{(1-\rho)^2}{1+(1-\rho)^2}$$

where $\rho = \max\{\rho_1, \rho_2\}$. This implies that

$$\ln G_T \leq \ln\left(\max\{S_T,1\}\right) - h(\rho, M)\sum_t r_t^2$$

where $h(\rho, M) = \frac{X(1-X)}{2(1+M)^2}$. The lemma follows since $\sum_t r_t^2 \geq Q_{min}$. □