

Testing Substitutability*

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Abstract

We provide an algorithm for testing the substitutability of a length- N preference relation over a set of contracts X in time $O(|X|^2 \cdot N^3)$. Access to the preference relation is essential for this result: We show that a substitutability-testing algorithm with access only to an agent's choice function must make an expected number of queries exponential in $|X|$. An analogous result obtains when the agent's preferences are quasilinear in a numeraire and the algorithm has access to the agent's underlying valuation function.

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1 Introduction

The theory of two-sided matching has been widely applied in medical labor markets such as the National Resident Matching Program (Roth and Peranson (1999)), the redesign of the gastroenterology fellowship market (Niederle and Roth (2003, 2005); McKinney et al. (2005)), and the job market for clinical psychologists (Roth and Xing (1997)).¹ The centralized matching mechanisms used in these markets require that hospitals submit *responsive* preferences over doctors, that is, preferences that render the choice between a pair of doctors independent of other available outcomes.² Although responsive preferences are easy to specify, they are unnatural in many settings: for instance, if a hospital desires both a cardiologist and a radiologist—and doctors of the same specialty substitute for one another—then that hospital’s preferences are not responsive.³

An alternative would be to allow hospitals to submit preferences that encode substitutabilities between doctors, such as those described above. This is possible within the domain of *substitutable* preferences introduced by Roth (1984); this domain is of particular interest, as in many settings it is the largest domain for which stable matching outcomes can be guaranteed.⁴ For the implementation of

¹Roth and Sotomayor (1990) and Roth (2008) survey the theory and practice of matching market design.

²More formally, a hospital’s preference relation is *responsive* if it is consistent with an ordinal ranking of doctors.

³For further discussion of substitutability, see Chapter 6 of Roth and Sotomayor (1990).

⁴Substitutability is necessary and sufficient for the existence of stable allocations in the settings of many-to-one matching (Roth (1984) proves sufficiency and Hatfield and Kojima (2008) prove necessity), many-to-many matching, (Roth (1984) and Echenique and Oviedo (2006) prove sufficiency and necessity follows from the results of Hatfield and Kojima (2008)), many-to-many matching with contracts (Klaus and Walzl (2009) and Hatfield and Kominers (2010a) prove sufficiency and Hatfield and Kominers (2010a) prove necessity), supply-chain matching (Ostrovsky (2008) and Hatfield

matching mechanisms with substitutable preferences, it is necessary that the market designer have a test which determines if a hospital’s submitted preference list is substitutable. In this paper, we provide such a test. While a naïve test requires time exponential in the number of options available to each hospital, we obtain a deterministic substitutability-testing algorithm that runs in time polynomial in the number of options and the preference list length.

Unfortunately, as we demonstrate, the efficient testing of substitutability depends crucially on the form in which agents’ preferences are submitted. In particular, we show that when the market designer only has oracle access to hospitals’ choice functions, substitutability can not be tested in subexponential time—even by randomized algorithms.

Classical matching theory typically assumes that agents only submit ordinal preference information, such as preference relations or choice functions.⁵ However, it is often appropriate to consider instead cardinal utility information, as in auction theory or exchange economies.⁶ Unfortunately, we find that the difficulty of testing substitutability given only choice function access carries over to the setting where the market designer can query valuation functions.

The remainder of this paper is organized as follows. In Section 2, we present our algorithm for testing the substitutability of a preference relation. In Sections 3

and Kominers (2010b) prove sufficiency and Hatfield and Kominers (2010b) prove necessity). In the setting of many-to-one matching with contracts, substitutable preferences are sufficient for the existence of a stable outcome (Hatfield and Milgrom (2005)), but are not necessary (Hatfield and Kojima (2008, 2010)).

⁵This is a standard assumption in many applications of matching, such as school choice, where monetary transfers among market participants are not allowed.

⁶As in matching theory, substitutability is essential in those settings. See, in particular, the work of Gul and Stacchetti (1999), Sun and Yang (2006), and Hatfield et al. (2010).

and 4, we show the difficulty of testing the substitutability of choice and valuation functions, respectively. Section 5 concludes.

2 Testing Substitutability of Preference Relations

We consider the preferences of a single agent and a finite set X of contracts available to that agent. The agent's preferences are specified by a preference relation \succ of the form

$$Y^1 \succ Y^2 \succ \dots \succ Y^N \succ \emptyset,$$

where $Y^n \subseteq X$ for each $n = 1, \dots, N$. This preference relation induces a choice function

$$C(Y) \equiv \max_{\succ} \{Z : Z \subseteq Y\}$$

on every set of contracts $Y \subseteq X$.⁷ Note that we suppress the dependence on the preference relation \succ in the notation for C , as throughout this section we will only consider a fixed preference relation \succ and its associated choice function.

Definition. The preference relation \succ is *substitutable* if for all $x, z \in X$ and $Y \subseteq X$, if $z \in C(\{x\} \cup Y \cup \{z\})$, then $z \in C(Y \cup \{z\})$. A tuple (x, z, Y) that fails this condition is called a *substitutability violation*.

Equivalently, if we define a rejection function

$$R(Y) \equiv Y - C(Y),$$

⁷Here, we use the notation \max_{\succ} to indicate that the maximization is taken with respect to \succ .

then the preference relation \succ is substitutable if and only if the rejection function R is isotone.⁸ Intuitively, this notion implies that no contracts x and z are complements: that is, substitutability guarantees that, having rejected the contract z , the agent never wishes to renege on that rejection when more contracts become available.

As stated, the substitutability condition imposes exponentially many conditions on the preference relation: one condition for each tuple (x, z, Y) . Thus, naïve substitutability-testing takes $O(|X|^2 \cdot 2^{|X|})$ time. Here we present an algorithm that tests preference relation substitutability in time polynomial in $|X|$, the number of contracts, and N , (defined as) the length of the submitted preference list.⁹

Intuitively, since the agent only chooses contract sets appearing in the preference list $Y^1 \succ \dots \succ Y^N \succ \emptyset$, one might think it only necessary to check for the violation of substitutability by tuples (x, z, Y^n) with $x, z \in \cup_m Y^m$. In fact this is not sufficient, as the following example shows.

Example 1. Let $X = \{a, b, b', c\}$ and let the preference relation be

$$\{a, b, c\} \succ \{b'\} \succ \{a, b\} \succ \{a, c\} \succ \{b, c\} \succ \{a\} \succ \{b\} \succ \{c\} \succ \emptyset.$$

These preferences are not substitutable, as $(a, c, \{b, b'\})$ is a substitutability violation.

However, no substitutability violation is of the form (x, z, Y^n) .

⁸Isotonicity of the rejection function can be stated in a fashion that makes substitutability violations apparent: R is isotone (equivalently, C is substitutable) if and only if for all $x, z \in X$ and $Y \subseteq X$, if $z \in C(Y)$, then $z \in C(Y - \{x\})$.

⁹Note that in some circumstances, N may be $O(2^{|X|})$, in which case our algorithm runs in time of the same order as that of the naïve algorithm. However, in many cases (such as when hospitals only intend to hire some small number of doctors from a large pool), our algorithm provides a significant improvement—a polynomial run time.

While it is sufficient to consider only (x, z, Y) with $x, z \in \cup_m Y^m$, it is not sufficient to check only $Y = Y^n$, as Y may need to contain multiple contracts that will only be chosen when another contract (x) is available. However, the intuition that one need only check sets Y derived from the sets Y^n is sound. In particular, as we show below, it suffices to check all tuples of the form $(x, z, (Y^m \cup Y^n) - \{x, z\})$. This gives rise to Algorithm 1, which in fact returns a list of all *minimal* substitutability violations, i.e. all violating tuples (x, z, Y) for which there does not exist $Y' \subsetneq Y$ such that (x, z, Y') is a substitutability violation.

Algorithm 1 *Tests the Substitutability of the Preference Relation \succ*

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1: set  $\mathcal{L} \leftarrow \emptyset$ 
2: for  $x, z \in \cup_\ell Y^\ell$  do
3:   for  $m = 1, \dots, N$  do
4:     if  $x, z \in Y^m$  then
5:       for  $n = m + 1, \dots, N$  do
6:         if  $x, z \notin Y^n$  then
7:            $Y \leftarrow (Y^m \cup Y^n) - \{x, z\}$ 
8:           if  $z \in C(\{x\} \cup Y \cup \{z\})$  and  $z \notin C(Y \cup \{z\})$  then
9:              $\mathcal{L} \leftarrow \mathcal{L} \cup \{(x, z, Y)\}$ 
10:          end if
11:        end if
12:      end for
13:    end if
14:  end for
15: end for
16: return  $\mathcal{L}$ 

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Algorithm 1 finds the (minimal) substitutability violation in Example 1 when checking contracts a and c in line 2. At that point, when $m = 1$ and $n = 2$, $Y^m = \{a, b, c\}$ and $Y^n = \{b'\}$, and the tuple (x, z, Y) is $(a, c, \{b, b'\})$.

To prove the efficacy of Algorithm 1, we make repeated use of the fact that the

choice function C induced by \succ has the property that removing unchosen contracts from the available set has no effect on the set of contracts chosen.¹⁰

Lemma 1. *For the choice function C induced by the preference relation $Y^1 \succ Y^2 \succ \dots \succ Y^N \succ \emptyset$, let n be such that $Y^n = C(Y)$ for $Y \subseteq X$. Then for any $Y' \subseteq Y$ such that $Y^n \subseteq Y'$, we have $Y^n = C(Y')$.*

Proof. Suppose $Y^m = C(Y')$. If $m < n$, then since Y^m is available from Y , it can not be the case that $Y^n = C(Y)$. We can not have that $m > n$, as Y^n is available from Y' . □

We now prove the correctness of Algorithm 1.

Theorem 1. *Algorithm 1 finds all minimal substitutability violations in a preference relation $Y^1 \succ \dots \succ Y^N \succ \emptyset$ in time $O(|X|^2 \cdot N^3)$.*

Proof. The time bounds are immediate from the statement of the algorithm, upon noting that computing $C(Z)$ for any $Z \subseteq X$ requires at most $O(N)$ time.¹¹ To show correctness, suppose there exists a set \hat{W} such that the tuple (x, z, \hat{W}) is a substitutability violation. Letting $W = \hat{W} - \{x, z\}$, we see that (x, z, W) is a substitutability violation as well, and if (x, z, \hat{W}) is minimal, then $W = \hat{W}$. Let $Y^m = C(W \cup \{x, z\})$ be the choice from options $W \cup \{x, z\}$ and $Y^n = C(W \cup \{z\})$ be the choice from options $W \cup \{z\}$.¹² Set $Y = (Y^m \cup Y^n) - \{x, z\}$. We will show that the tuple (x, z, Y) is a substitutability violation.

¹⁰This is analogous to the notion of independence of irrelevant alternatives in revealed preference theory.

¹¹We assume it takes constant time to check the subset relation.

¹²Note that $x, z \notin Y^n$, as $x, z \notin W$ and (x, z, W) is a substitutability violation. Thus, without loss of functionality (and a slight gain in speed) we may include line 6 in Algorithm 1.

- First note that as (x, z, W) is a substitutability violation, $z \in Y^m = C(W \cup \{x, z\})$. Furthermore, $Y \cup \{x, z\} \subseteq W \cup \{x, z\}$ and $C(W \cup \{x, z\}) = Y^m \subseteq Y \cup \{x, z\}$ so, by Lemma 1, $C(Y \cup \{x, z\}) = C(W \cup \{x, z\}) = Y^m$. Thus, $z \in C(Y \cup \{x, z\})$.
- Next note that as (x, z, W) is a substitutability violation, we also have $z \notin Y^n = C(Y \cup \{z\})$. Again, $Y \cup \{z\} \subseteq W \cup \{z\}$ and $C(W \cup \{z\}) = Y^n \subseteq Y \cup \{z\}$ so, by Lemma 1, $C(Y \cup \{z\}) = C(W \cup \{z\}) = Y^n$. Thus, $z \notin C(Y \cup \{z\})$.

This shows that (x, z, Y) is a substitutability violation.

To see that the algorithm finds all minimal violations, simply note that for any substitutability violation (x, z, W) , we know that for the returned tuple (x, z, Y) , we have that $Y \subseteq W$ and (x, z, Y) is a violation. Thus if (x, z, W) is minimal, it must be that $Y = W$. \square

Given the list of minimal substitutability violations, it is easy to check (in constant time for each violation) whether these violations constitute violations of the weaker *bilateral* and *unilateral substitutes* conditions introduced by Hatfield and Kojima (2010). Thus, Algorithm 1 can easily be extended to test for bilateral or unilateral substitutability in time polynomial in $|X|$ and N .

3 Testing Substitutability of Choice Functions

We now consider algorithms which have oracle access to the agent's choice function over contracts, C , but do not have access to the preference relation itself. We show

that no algorithm can verify the choice function’s substitutability in subexponential time. The key observation underlying this result is that a substitutability violation is “local”—it is possible for a choice function to have very few substitutability violations relative to the number of sets of contracts.

Our proof uses the minimax principle of Yao (1977), which states that the expected running time of the best randomized algorithm on a worst-case deterministic input is equal to the expected running time of the best deterministic algorithm on the worst-case probability distribution of inputs.¹³

More precisely, for a problem with deterministic inputs \mathcal{I} and deterministic algorithms \mathcal{A} , let $\Delta(\mathcal{I})$ denote the set of probability distributions over \mathcal{I} and $\Delta(\mathcal{A})$ denote the set of probability distributions over \mathcal{A} . For $A \in \mathcal{A}$ and $I \in \mathcal{I}$, let $A(I)$ denote the running time of A on I .

Theorem 2 (Yao (1977)). *For a problem with deterministic inputs \mathcal{I} and deterministic algorithms \mathcal{A} ,*

$$\min_{\delta_{\mathcal{A}} \in \Delta(\mathcal{A})} \max_{I \in \mathcal{I}} E_{A \sim \delta_{\mathcal{A}}}[A(I)] = \max_{\delta_{\mathcal{I}} \in \Delta(\mathcal{I})} \min_{A \in \mathcal{A}} E_{I \sim \delta_{\mathcal{I}}}[A(I)].$$

The proof of Theorem 2 follows from von Neumann’s minimax theorem for two-player, zero-sum games. Here, the two players are the algorithm designer and the adversary. The algorithm designer’s strategy set is all possible deterministic algorithms \mathcal{A} ; the adversary’s strategy set is all possible deterministic inputs \mathcal{I} . The payoff of a pure strategy profile (A, I) to the adversary is the running time $A(I)$ (the

¹³Here, by a *randomized* algorithm, we mean a probability distribution over deterministic algorithms.

payoff to the algorithm designer is the negative of this as we are defining a zero-sum game). For further details, we refer the reader to Yao (1977).

We now apply Theorem 2 to show that a substitutability-testing algorithm with only oracle access to C must make exponentially many queries in expectation. For any $Y \subsetneq X$ with $|X - Y| \geq 2$, we define the following choice function:¹⁴

$$C_Y(Z) = \begin{cases} Y & Y \subseteq Z \text{ and } |Z| = |Y| + 1, \\ Z & \text{otherwise.} \end{cases}$$

We note that C_Y is not substitutable: for any $x, z \notin Y$, we have $z \notin Y = C_Y(Y \cup \{z\})$, but $z \in \{x\} \cup Y \cup \{z\} = C_Y(\{x\} \cup Y \cup \{z\})$.

Now we consider any deterministic substitutability testing algorithm A and suppose that the distribution of inputs is as follows. With probability $\frac{1}{2}$, set $C = \text{id}$, where id is the *identity function* on X , i.e. $\text{id}(Z) = Z$ for all $Z \subseteq X$. With the remaining probability, select a set Y of cardinality $q \equiv \lfloor \frac{|X|}{2} \rfloor$ uniformly at random and set $C = C_Y$.

We observe that A must query at least

$$\frac{1}{q+1} \binom{|X|}{q} \tag{1}$$

sets in order to be sure that no substitutability violation exists: Since the functions id and C_Y only differ on input sets $Z \supsetneq Y$ of size $|Z| = |Y| + 1$, only queries on such sets Z can distinguish them. Thus, verifying that $C = \text{id}$ requires distinguishing C

¹⁴We implicitly assume $|X| > 3$.

from $\binom{|X|}{q}$ possible assignments of C_Y . Querying C on a given set Z of size $q + 1$ simultaneously tests whether $C = C_{Z-\{z\}}$ for each $z \in Z$, checking a total of $q + 1$ possible assignments of C_Y . Hence, the algorithm A must check at least as many sets of size $q + 1$ as in (1).

Theorem 3. *The worst-case expected running time of any randomized algorithm that verifies substitutability given only oracle access to choice functions is at least exponential in $|X|$, the number of contracts.*

Combining Theorems 2 and 3, we obtain the following corollary.

Corollary 1. *The worst-case running time of any deterministic algorithm that verifies substitutability given only oracle access to choice functions is at least exponential in $|X|$, the number of contracts.*

We note that even allowing polynomial error in the algorithm does not improve the running time substantially, i.e. there is no subexponential-time algorithm that verifies substitutability correctly with probability $1 - o(1)$.

4 Testing Substitutability of Valuation Functions

We now consider the case where each contract defines the transfer of an item at an associated price; let the set of items be denoted Ω , and let the price of $\omega \in \Omega$ be denoted p_ω . For many applications (e.g., Kelso and Crawford (1982); Gul and Stacchetti (1999, 2000); Sun and Yang (2006, 2009); Hatfield et al. (2010)), it is

assumed that agents' utility functions take the quasilinear form

$$U(\Psi, p) = v(\Psi) - \sum_{\psi \in \Psi} p_{\psi},$$

for $\Psi \subseteq \Omega$.

From here, we can define the demand correspondence $D : \mathbb{R}^{|\Omega|} \rightarrow \mathcal{P}(\Omega)$,

$$D(p) \equiv \arg \max_{\Psi \subseteq \Omega} U(\Psi, p).$$

Kelso and Crawford (1982) introduced the following natural notion of substitutability.

Definition. A demand correspondence D satisfies the *gross substitutes* condition if for every profile of prices p , every item $\psi \in \cup_{\Psi \in D(p)} \Psi$, and all price vectors \hat{p} such that $\hat{p} \geq p$ and $\hat{p}_{\psi} = p_{\psi}$, we have $\psi \in \cup_{\Psi \in D(\hat{p})} \Psi$.

The gross substitutes condition has been shown by Hatfield et al. (2010) to be essential for the guaranteed existence of stable allocations and competitive equilibria in a variety of settings. Additionally, the gross substitutes condition has been rephrased in terms of conditions on the valuation function v .

Theorem 4 (Reijnierse et al. (2002); Fujishige and Yang (2003)). *A demand correspondence D satisfies the gross substitutes condition if and only if the corresponding valuation function v is M^{\natural} -concave, i.e. for each $\Psi \subseteq \Omega$:*

1. $v(\Psi \cup \{\psi, \chi\}) + v(\Psi) \leq v(\Psi \cup \{\psi\}) + v(\Psi \cup \{\chi\})$ for all $\psi, \chi \notin \Psi$, and

2. $v(\Psi \cup \{\psi, \chi\}) + v(\Psi \cup \{\varphi\}) \leq \max\{v(\Psi \cup \{\psi, \varphi\}) + v(\Psi \cup \{\chi\}), v(\Psi \cup \{\chi, \varphi\}) + v(\Psi \cup \{\psi\})\}$ for all $\psi, \chi, \varphi \notin \Psi$.

As Reijnierse et al. (2002) point out, this theorem implies an algorithm that tests gross substitutability and runs in time $O(|X|^3 \cdot 2^{|X|})$. We now show that this algorithm is optimal up to polynomial factors; i.e. there is no subexponential-time algorithm that verifies gross substitutability given oracle access to the valuation function.¹⁵

Theorem 5. *The worst-case expected running time of any randomized algorithm verifying gross substitutability given only oracle access to the valuation function is at least exponential in $|\Omega|$, the number of items.*

Corollary 2. *The worst-case running time of any deterministic algorithm verifying gross substitutability given only oracle access to the valuation function is at least exponential in $|\Omega|$, the number of items.*

The proof of these results follows the same strategy as that used to prove Theorem 3 and Corollary 1. For any $\Psi \subsetneq \Omega$ with $|\Omega - \Psi| \geq 2$, we define the following valuation function:

$$v_{\Psi}(\Phi) = \begin{cases} |\Phi| + \frac{1}{2} & \Phi = \Psi, \\ |\Phi| & \text{otherwise.} \end{cases}$$

¹⁵This result is not an exact analogue of Theorem 3, as it involve queries to valuation functions rather than demand correspondences. The complexity of testing gross substitutability given oracle access to the demand correspondence is presently unclear. The difficulty involved in proving an exact analogue of Theorem 3 arises from the fact that demand correspondences are functions of prices, rather than sets of items.

Note that v_Ψ is not M^{\natural} -concave since for any $\chi, \varphi \notin \Psi$,

$$v_\Psi(\Psi \cup \{\varphi, \chi\}) + v_\Psi(\Psi) = 2|\Psi| + 2 + \frac{1}{2} > 2|\Psi| + 2 = v_\Psi(\Psi \cup \{\varphi\}) + v_\Psi(\Psi \cup \{\chi\}).$$

As in the proof of Theorem 3, distinguishing v_Ψ from the valuation \hat{v} defined by $\hat{v}(\Phi) = |\Phi|$ requires exponentially many queries in expectation. Moreover, as in the case of Theorem 3 and Corollary 1, allowing polynomial error in the algorithm does not improve the running time significantly.

5 Conclusion

We have shown a method for testing the substitutability of an agent's preferences relations in time polynomial in the length of the preference relation. Since agents are, in general, only able to construct short (i.e. polynomial length) preferences lists, our algorithm is an effective method for the market designer to test whether the submitted preferences are substitutable. In principle, such an algorithm could be distributed to market participants for use in the preparation of their preference relations for submission.

Unfortunately, testing substitutability with only oracle access to the choice function is far more difficult—it requires exponential time in general. The problem persists even for the special case in which preferences are quasilinear in a numeraire and only valuation function access is available.

Our results imply that market designers should develop specific methods to transmit large classes of substitutable preferences. Milgrom (2009) provides one such

method, assignment messages, for environments such as electricity and commodity markets where preferences are quasilinear. For other settings, it is likely that alternative languages for expressing substitutable preferences are needed. It is unclear how general such a language can be while remaining effective, but our results do not rule out the existence of an encoding under which all substitutable preferences take a form of at most polynomial size.

References

- Echenique, F. and J. Oviedo (2006). A theory of stability in many-to-many matching markets. *Theoretical Economics* 1, 233–273.
- Fujishige, S. and Z. Yang (2003). A note on Kelso and Crawford’s gross substitutes condition. *Mathematics of Operations Research*, 463–469.
- Gul, F. and E. Stacchetti (1999). Walrasian equilibrium with gross substitutes. *Journal of Economic Theory* 87, 95–124.
- Gul, F. and E. Stacchetti (2000). The English auction with differentiated commodities. *Journal of Economic Theory* 92, 66–95.
- Hatfield, J. W. and F. Kojima (2008). Matching with contracts: Comment. *American Economic Review* 98, 1189–1194.
- Hatfield, J. W. and F. Kojima (2010). Substitutes and stability for matching with contracts. *Journal of Economic Theory* 145(5), 1704–1723.

- Hatfield, J. W. and S. D. Kominers (2010a). Contract design and stability in matching markets. Mimeo, Harvard Business School.
- Hatfield, J. W. and S. D. Kominers (2010b). Matching in Networks with Bilateral Contracts. Stanford Graduate School of Business Working Paper 2050.
- Hatfield, J. W., S. D. Kominers, A. Nichifor, M. Ostrovsky, and A. Westkamp (2010). Stability and competitive equilibrium in trading networks. Mimeo, Stanford Graduate School of Business.
- Hatfield, J. W. and P. Milgrom (2005). Matching with contracts. *American Economic Review* 95, 913–935.
- Kelso, A. S. and V. P. Crawford (1982). Job matching, coalition formation, and gross substitutes. *Econometrica* 50, 1483–1504.
- Klaus, B. and M. Walzl (2009). Stable many-to-many matchings with contracts. *Journal of Mathematical Economics* 45(7-8), 422–434.
- McKinney, C. N., M. Niederle, and A. E. Roth (2005). The collapse of a medical labor clearinghouse (and why such failures are rare). *American Economic Review* 95, 878–889.
- Milgrom, P. (2009). Assignment messages and exchanges. *American Economic Journal: Microeconomics* 1, 95–113.
- Niederle, M. and A. E. Roth (2003). Unraveling reduces mobility in a labor market: Gastroenterology with and without a centralized match. *Journal of Political Economy* 111, 1342–1352.

- Niederle, M. and A. E. Roth (2005). The gastroenterology fellowship market: Should there be a match? *American Economic Review* 95, 372–375.
- Ostrovsky, M. (2008). Stability in supply chain networks. *American Economic Review* 98, 897–923.
- Reijnierse, H., A. van Gellekom, and J. Potters (2002). Verifying gross substitutability. *Economic Theory* 20(4), 767–776.
- Roth, A. E. (1984). Stability and polarization of interests in job matching. *Econometrica* 52, 47–57.
- Roth, A. E. (2008). Deferred acceptance algorithms: history, theory, practice, and open questions. *International Journal of Game Theory* 36, 537–569.
- Roth, A. E. and E. Peranson (1999). The effects of the change in the NRMP matching algorithm. *American Economic Review* 89, 748–780.
- Roth, A. E. and M. A. O. Sotomayor (1990). *Two-sided matching: a study in game-theoretic modeling and analysis*. Econometric Society Monographs. Cambridge University Press.
- Roth, A. E. and X. Xing (1997). Turnaround time and bottlenecks in market clearing: Decentralized matching in the market for clinical psychologists. *Journal of Political Economy* 105, 284–329.
- Sun, N. and Z. Yang (2006). Equilibria and indivisibilities: gross substitutes and complements. *Econometrica* 74, 1385–1402.

- Sun, N. and Z. Yang (2009). A double-track adjustment process for discrete markets with substitutes and complements. *Econometrica* 77, 933–952.
- Yao, A. C.-C. (1977). Probabilistic computations: Toward a unified measure of complexity. In *Proceedings of the 18th Annual Symposium on Foundations of Computer Science (FOCS)*, pp. 222–227.