1. Problems 1.5, 1.7 and 1.8 from Gibbons.

**Gibbons #1.5**

The question asks you to express the Cournot duopoly game as a Prisoners’ Dilemma where the only two available quantities are the monopoly quantity, $q_m$, and the Cournot equilibrium quantity $q_c$. To do this, you will need to calculate the payoffs to each player under each combination of these strategies. That is for each player $i$ we know that the profit function is

$$\pi_i(q_i, q_j) = q_i[a - c - (q_i + q_j)]$$

So for player 1, the following values must be calculated.

- $\pi_m = \pi_1(q_m/2, q_m/2) = (a - c)^2 / 8$
- $\pi_c = \pi_1(q_c, q_c) = (a - c)^2 / 9$
- $\pi_t = \pi_1(q_c, q_m/2) = 5(a - c)^2 / 36$
- $\pi_s = \pi_1(q_m/2, q_c) = 5(a - c)^2 / 48$

It can be verified that $\pi_t > \pi_m > \pi_c > \pi_s$ which corresponds to the payoffs for the Prisoners’ Dilemma game. These payoffs and strategies can be represented in matrix form as below.

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<tr>
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<th>$q_m$</th>
<th>$q_c$</th>
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<tbody>
<tr>
<td>$q_m$</td>
<td>$\pi_m, \pi_m$</td>
<td>$\pi_s, \pi_t$</td>
</tr>
<tr>
<td>$q_c$</td>
<td>$\pi_t, \pi_s$</td>
<td>$\pi_c, \pi_c$</td>
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The second part of the questions asks you to extend the game by adding a third possible quantity, $q'$. The game must be such that the Cournot output is still the unique Nash equilibrium, and there are no strictly dominant strategies. There is no single solution to this problem, but one way to proceed is as follows. First set up a hypothetical matrix of this game as below.
where

\[ A = \pi_1\left(\frac{q_m}{2}, q'\right) \]
\[ B = \pi_1(q', \frac{q_m}{2}) \]
\[ C = \pi_1(q_c, q') \]
\[ D = \pi_1(q', q_c) \]
\[ E = \pi_1(q', q') \]

Next, note that in order for \((q_c, q_c)\) to be a Nash equilibrium, \(\pi_c > D\). Then, in order for \((q_c, q_c)\) to be a unique Nash equilibrium \(q'\) cannot be a best response for either player, when the other is playing \(q'\) [otherwise \((q', q')\) is also a NE]. So either \(A > E\) or \(C > E\) or both.

For there to be no strictly dominant strategies, \(q'\) must either be a best response for one of the other strategies, or allow \(\frac{q_m}{2}\) to be a best response to \(q'\). Let us assume that \(\frac{q_m}{2}\) is a best response to \(q'\) for both players. Then \(A > C > E\). Consequently, for \((q_c, q_c)\) to be a unique Nash equilibrium it must be the case that \(B < \pi_t\). Using this information, we can now calculate a value for \(q'\) and then verify that it satisfies all the requirements above.

The best response function for firm 1 in the Cournot duopoly game is \(R_1(q_2) = \frac{1}{2}(a - c - q_2)\). So for \(\frac{q_m}{2}\) to be a best response to \(q'\), \(q'\) must satisfy

\[ R_1(q') = \frac{q_m}{2} = \frac{1}{2}(a - c - q') \]

This yields \(q' = \frac{a-c}{2}\), which implies

\[ A = \pi_1\left(\frac{q_m}{2}, q'\right) = \frac{(a-c)^2}{16} \]
\[ B = \pi_1(q', \frac{q_m}{2}) = \frac{(a-c)^2}{8} \]
\[ C = \pi_1(q_c, q') = \frac{(a-c)^2}{18} \]
\[ D = \pi_1(q', q_c) = \frac{(a-c)^2}{12} \]
\[ E = \pi_1(q', q') = 0 \]

All of which satisfy the requirements above.
Firm $i$’s payoff function is

$$
\pi_i(p_i, p_j) = \begin{cases} 
(p_i - c)(a - p_i) & \text{if } p_i < p_j, \\
\frac{1}{2}(p_i - c)(a - p_i) & \text{if } p_i = p_j \text{ for } i \neq j, \\
0 & \text{if } p_i > p_j,
\end{cases}
$$

To show that $p^*_1 = p^*_2 = c$ is the unique Nash equilibrium, rule out all other possibilities. First, neither firm will charge a price below $c$. If $p_i \leq p_j$ and $p_i < c$, then $\Pi_i < 0$, and firm $i$ is better off if it raises its price. Second, if both firms’ prices are above $c$, then the firm with the higher price wants to undercut its rival. That is, if $c < p_i \leq p_j$, then firm $j$ would increase its profits by setting its price equal to $p_i - \epsilon$. Third, if one firm’s price equals $c$ while the other firm charges a price above $c$, then the firm charging $c$ wants to increase its price. That is, if $c = p_i < p_j$, then firm $i$ wants to increase its price to $p_j - \epsilon$. The only possibility left is $p_1 = p_2 = c$. At this point, neither firm has any incentive to deviate, and we have our Nash equilibrium.

Gibbons #1.8

You are given a population of voters, uniformly distributed over the ideological spectrum, which you can take to be the interval $[0,1]$. Strategies (“platforms”) are points in $[0,1]$, and payoffs are probabilities of being elected.

Consider first the model with two candidates. Let $x_1$ denote candidate 1’s strategy and $x_2$ denote candidate 2’s strategy. The pure-strategy Nash equilibrium is $x^*_1 = x^*_2 = \frac{1}{2}$. At $x^*_1 = x^*_2 = \frac{1}{2}$, each candidate wins the election with probability $\frac{1}{2}$. Any deviation from this strategy combination causes the deviator to lose the election.

This ideological blandness is the unique pure-strategy Nash equilibrium. To show this, consider the other possibilities, $x^*_1 = x^*_2 = x^* \neq \frac{1}{2}$ and $x^*_1 \neq x^*_2$. You should be able to convince yourself that neither of these possibilities constitutes a Nash equilibrium. In both cases, one or both of the candidates would have an incentive to deviate.

The case with three candidates is more complicated – there is an infinity of pure-strategy Nash equilibria. Consider $x^*_1 = .3$, $x^*_2 = .4$, $x^*_3 = .6$. Candidate 3 gets 50% of the vote ($\frac{1}{2} = (1 - .6) + \frac{.5 - .4}{2}$) and wins the election. She obviously has no incentive to deviate. Candidate 2, even though she gets only 15% of the vote and loses the election, also has no incentive to deviate. It is true that she can increase her vote total by changing position but, given the positions of the other two candidates, there is no move she can make that will give her any chance of winning the election. Similarly, candidate 1 has no incentive to deviate. Thus, $x^*_1 = .3$, $x^*_2 = .4$, $x^*_3 = .6$ is a pure-strategy Nash equilibrium. You should be able to find many, many more.
Find all the Nash equilibria of the following game.

This game has one pure strategy Nash Equilibrium \((B, R)\).

To check if this game has any mixed strategy equilibrium, start by assuming that player 1 uses the mixed strategy \(p_1 = (p_{11}, p_{12}, 1 - p_{11} - p_{12})\). For player 2 to randomize, he (she) must be indifferent between playing \(L, C\) and \(R\). Hence player 2’s expected payoff playing all three must be equal. That is,

\[
   p_{11} \cdot 0 + p_{12} \cdot 3 + (1 - p_{11} - p_{12}) \cdot 0 = p_{11} \cdot 3 + p_{12} \cdot 0 + (1 - p_{11} - p_{12}) \cdot 0 = p_{11} \cdot 2 + p_{12} \cdot 0 + (1 - p_{11} - p_{12}) \cdot 0
\]

Comparing expressions (1), (2) and (3), it is obvious that the expected payoff playing \(R\) is strictly less than playing both \(L\) and \(C\). That means that there are no beliefs about player 1’s strategies that player 2 could have, that would cause player 2 to play \(R\) with positive probability.

By a similar reasoning, we can show that player 1 will never place a positive probability on playing \(B\). This means that \(1 - p_{11} - p_{12} = 0\) or \(p_{11} + p_{12} = 1\). Therefore, we need to calculate \(p_{11}\) and \(p_{12}\) such that \(p_{12} \cdot 3 = p_{11} \cdot 3\) and \(p_{11} + p_{12} = 1\), which gives \(p_{11} = p_{12} = \frac{1}{2}\).

By the same calculation for player 2 we can calculate that \(p_{21} = p_{22} = \frac{1}{2}\), where \(p_{21}\) is the probability that player 2 places on \(L\), and \(p_{22}\) is the probability that player 2 places on \(C\).

The mixed strategy Nash equilibrium for this game is therefore \((p_1^*, p_2^*)\) such that \(p_1^* = (\frac{1}{2}, \frac{1}{2}, 0)\) and \(p_2^* = (\frac{1}{2}, \frac{1}{2}, 0)\).

(a) Which strategies survive the process of iterated deletion of strictly dominated strategies?

All strategies survive iterated deletion of strictly dominated strategies.
(b) Now suppose that we consider the possibility that a pure strategy may be dominated by a mixed strategy. How does your answer to part (a) change in this case (if at all)? Explain.

The mixed strategy \( p_1 = \left( \frac{1}{2}, \frac{1}{2}, 0 \right) \) strictly dominates playing \( B \) for player 1. After elimination of \( B \), the mixed strategy \( p_2 = \left( \frac{1}{2}, \frac{1}{2}, 0 \right) \) strictly dominates playing \( R \) for player 2. Therefore when considering mixed strategies, the strategies that survive iterated deletion of strictly dominated strategies are \( (T, M) \) for player 1 and \( (L, C) \) for player 2.

4. Consider a Cournot game with \( n \) identical firms and a linear inverse demand function \( p(Q) = \alpha - \beta Q \) where \( Q = \sum_{i=1}^{n} q_i \) is total industry output. Each firm has constant marginal cost \( c \) of producing output.

(a) Solve for the unique Nash equilibrium of the game.

When you derive the best response function for each firm \( i \) you should get:

\[
R_i(q_{-i}) = \frac{\alpha - c}{2\beta} - \frac{1}{2} \sum_{j \neq i} q_j
\]

Exploiting symmetry between the firms you can set \( q_i = q^* \) for all \( i \), and solving for \( q^* \) you obtain \( q^* = \frac{\alpha - c}{(n+1)\beta} \) for each firm \( i \) which is the unique NE.

(b) How much profit does each firm make in equilibrium? What happens to total industry profits as \( n \) increases?

Substituting \( q^* = \frac{\alpha - c}{(n+1)\beta} \) into the profit function, you should get that each firm makes a profit of

\[
\pi_i = \frac{(\alpha - c)^2}{(n+1)^2\beta}
\]

The total industry profit is then

\[
\Pi = n \frac{(\alpha - c)^2}{(n+1)^2\beta}
\]

Industry profit goes to zero as \( n \) goes to infinity.

(c) Assume that there are three firms \( (n = 3) \). Show that all firms would want to merge to form a monopoly.

Before the merger each firm receives profits of

\[
\pi_i = \frac{(\alpha - c)^2}{16\beta}
\]

After the merger, the profits of the joint firm would be \( \pi_i = \frac{(\alpha - c)^2}{4\beta} \) a third of which equals
\[ \pi_i = \frac{(\alpha - c)^2}{12\beta} \]

This is greater than the profits of each firm acting independently, so they would like to merge.

(d) Assume again that there are three firms playing a Cournot game. The FTC prohibits monopoly in this industry but is worried about a merger of two firms such that the industry would become a duopoly. Show that the FTC’s worries are unfounded.

Prior to a merge the two firms would have joint profits of

\[ \pi_i = \frac{(\alpha - c)^2}{8\beta} \]

However, the merged firm cannot commit to keep output at the pre-merger level because it is off its best response curve. It would do better by decreasing output slightly to increase prices. The remaining firm realizes this and increases its output slightly. This will tempt the merged firm to decrease output even more etc. Eventually we are back at the 2-firm Cournot solution which will give the merged firm a profit of

\[ \pi_i = \frac{(\alpha - c)^2}{9\beta} \]

Which is lower than their pre-merger profits.

5. Two employees work together in a team. Worker \( i = 1, 2 \) contributes some positive effort \( e_i \) to the team output \( y = e_1 + e_2 \) (the output is measured in dollar terms) which is equally divided up between both workers. This effort causes the worker a disutility of \( \frac{1}{2}e_i^2 \).

(a) Assume that you are a social planner whose aim is to maximize the total utility of both workers (i.e. the sum of their utilities). How much effort should both workers choose?

The social planner will choose effort levels to maximize social welfare. Social welfare, is the joint welfare of both players which is given by

\[ u_1 + u_2 = e_1 + e_2 - \frac{1}{2}e_1^2 - \frac{1}{2}e_2^2 \]

A first order condition with respect to each player’s effort level will yield \( e_1 = e_2 = 1 \).

(b) Find the pure strategy Nash equilibrium of the game. Compare your result to the social planner’s solution.

Each player will respond to its own private incentives. Each player gets utility

\[ u_i = \frac{1}{2}[e_1 + e_2] - \frac{1}{2}e_i^2 \]
for \( i = 1, 2 \). Therefore each of them has a best response function of \( e_i = \frac{1}{2} \) regardless of the other player’s effort level. This is the pure strategy Nash equilibrium.

The Nash equilibrium is clearly suboptimal. The problem is that players do not enjoy the full social benefit of their actions. If they exert more effort \( \Delta e_i \) they generate additional social surplus \( \Delta e_i \) but they themselves get only half of it. Therefore, the incentive to work hard is reduced and since the marginal cost of exerting effort increases with \( e_i \) (due to convex effort costs) workers will exert less effort than is socially optimal.

6. Consider the following game-theoretic model of the equilibrium determination of the cleanliness (and effort distribution) of an apartment shared by two roommates. In the game, the two roommates simultaneously choose the effort, \( e_1 \) and \( e_2 \), to spend on apartment cleaning. They each get utility from the cleanliness of the apartment (which is a function of the sum of the efforts) and disutility from the effort they personally expend. Player 1 places a higher valuation on cleanliness. Specifically, assume that \( e_1 \) and \( e_2 \) are chosen from the set of nonnegative real numbers and that

\[
\begin{align*}
  u_1(e_1, e_2) &= k \log(e_1 + e_2) - e_1 \\
  u_2(e_1, e_2) &= \log(e_1 + e_2) - e_2
\end{align*}
\]

where \( k > 1 \).

(a) Find the two players’ best response functions.

In this problem it is important to specify the best response function for all effort levels of the other player:

\[
\begin{align*}
  R_1(e_2) &= \begin{cases} 
  k - e_2 & \text{for } 0 \leq e_2 \leq k, \\
  0 & \text{for } e_2 > k
  \end{cases} \\
  R_2(e_1) &= \begin{cases} 
  1 - e_1 & \text{for } 0 \leq e_1 \leq 1, \\
  0 & \text{for } e_1 > 1
  \end{cases}
\end{align*}
\]

(b) Find the pure strategy Nash equilibrium of the game. How does the equilibrium distribution of effort reflect the differences in the players’ tastes. *Hint: Your BR function should be defined for each positive effort level and has to be non-negative everywhere.*

A graphical solution for this Nash equilibrium is the simplest. The graph below represents the two best response functions.
The dot indicates the unique intersection of the two best response functions at $e^*_1 = k$ and $e^*_2 = 0$. This is the unique Nash equilibrium of the game. In this Nash equilibrium the person who values cleanliness more does all the work.

(c) Try to write down a modification of the model above in which the outcome seems more fair.

There is no single answer to this question. A possible answer would modify the utility functions so that they are

$$u_1(e_1, e_2) = k \log(e_1) + \log(e_2) - e_1$$
$$u_2(e_1, e_2) = \log(e_1) + \log(e_2) - e_2$$

In this case, $e^*_1 = k$ and $e^*_2 = 1$. That is, both players exert some effort.